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Mathematics Teaching in the Middle School

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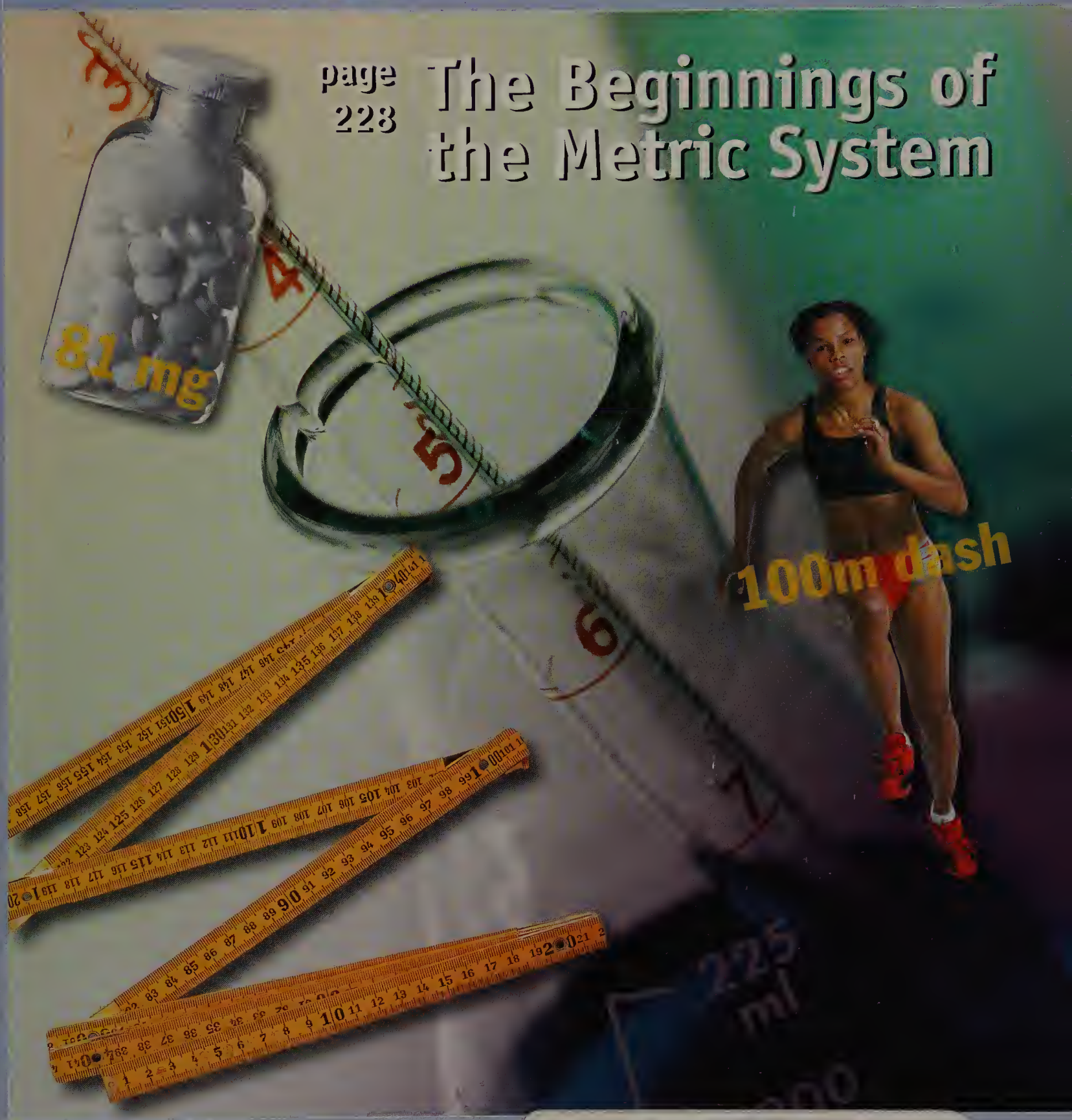
The Beginnings of the Metric System



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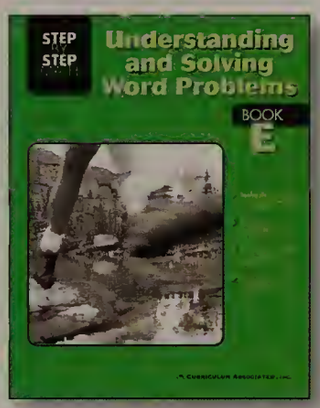

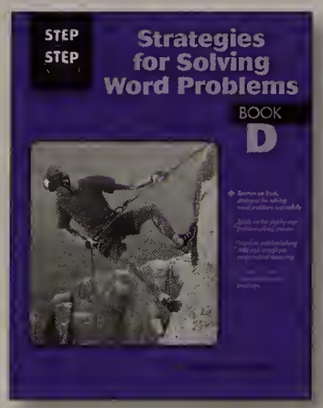

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ON THE COVER

The items and elements pictured on the cover are just a few of the metric measurements that may have originated about 400 years ago. Read "Math Roots: The Beginnings of the Metric System" by authors Art Johnson and Kit Norris to learn the history behind the mathematics.

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Mathematics Teaching in the Middle School

All correspondence should be addressed to *Mathematics Teaching in the Middle School*, 1906 Association Drive, Reston, VA 20191-1502. Manuscripts should be prepared according to the *Chicago Manual of Style* and the United States Metric Association's *Guide to the Use of the Metric System*. No author identification should appear on the manuscript; the journal uses a blind-review process. To submit a manuscript, access **mtms.msubmit.net**.

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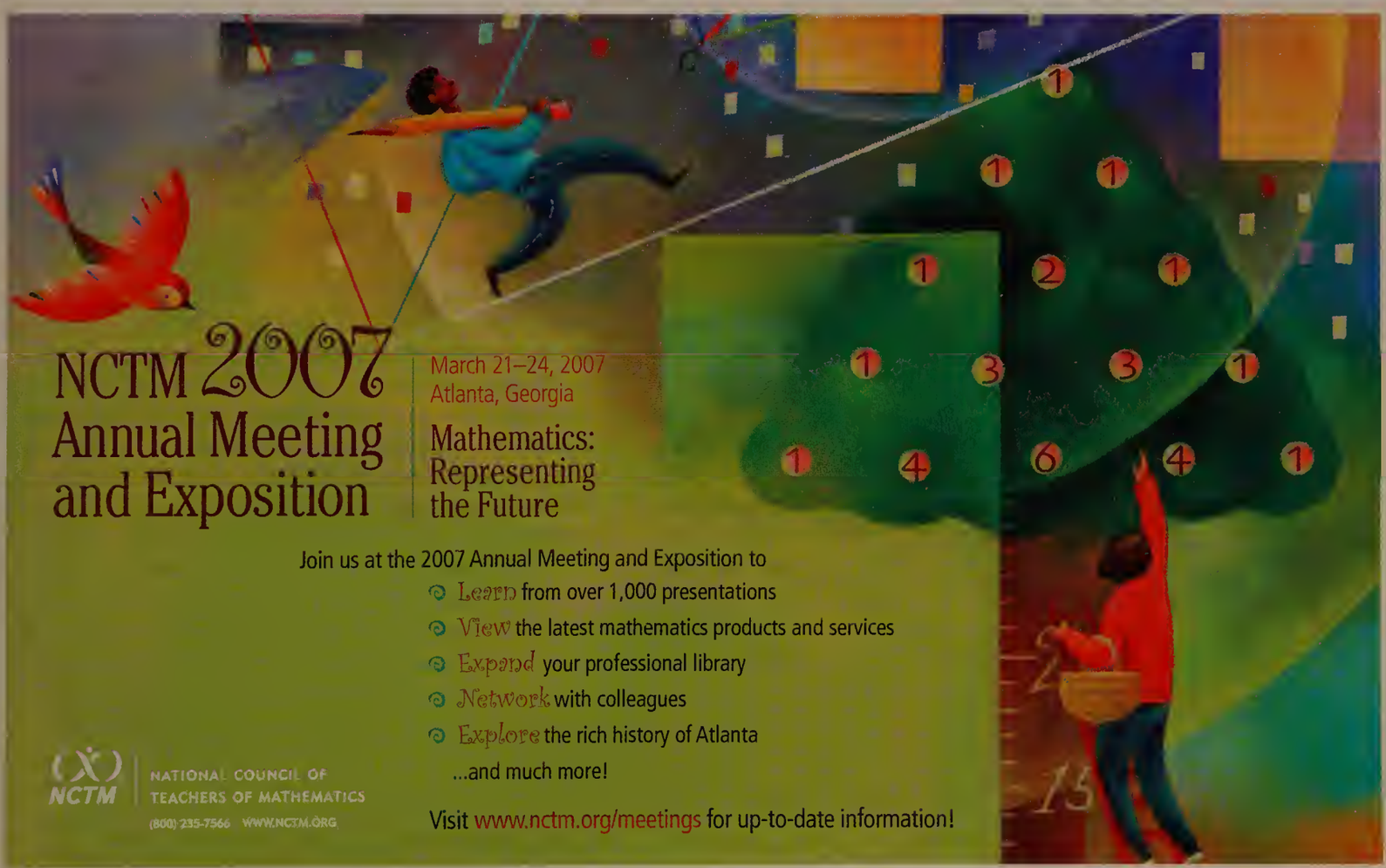

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MATH ROOTS

ART JOHNSON AND KIT NORRIS

The Beginnings of the Metric System

THE METRIC SYSTEM IS SLOWLY BUT surely becoming an integral part of American society. Two-liter bottles of soda, 100-meter dashes, and milligrams of medicine are now commonplace. It is more important than ever that students have a working knowledge of the metric system. When students study metrics in mathematics class, it can be beneficial for them to learn something about its origins. The historical underpinnings of the metric system can be instructive and entertaining for students in the middle grades.

The advent of the metric system extends back about 400 years. French mathematician Simon Stevin (1548–1620) published *De Thiende*, in which he advocated the use of decimal fractions as an efficient way to represent rational numbers. The pamphlet also suggested a measurement system based on powers of ten, but nothing resulted from it. Gabriel Mouton, the vicar of St. Paul's Church in Lyon, France, took up the metric cause about 100 years later. The basic unit in Mouton's system was one minute of the length of a line of longitude, about 34 meters, with subdivisions based on powers of ten. In 1675, Tito Livio Burattini proposed a *meter* as the standard unit of length for the world community. He envisioned a time when "all civilized people on Earth would use the same weights and measures despite differences in language and customs" (Strauss 1995).

The matter of a new measurement system lay dormant until 1789. In France at that time, over 800 different measures were in use. There were thirteen distinct measures for feet, ranging from 10.6 inches to 13.4 inches; twelve different pounds ranging from 0.758 to 1.143 pounds; and twenty-three different bushels. Some measurements, such as the *toise* (1.949 m), were unique to certain districts. With respect to measurement, France was not unique; most European countries contained provinces, districts, and even towns that used unique combinations of measures designed to meet local needs. In most countries, and in France in particular, there were continual calls to reform the confusing systems of weights and measures. The central government in France finally took steps to improve matters, but it took a revolution of historic proportions.

The French Revolution changed everything. While removing the monarchy, revolutionaries eradicated anything that smacked of the old regime. City names, clothing fashions, and even the calen-

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Edited by THOMASENIA LOTT ADAMS, *tla@coe.ufl.edu, who teaches in the College of Education at the University of Florida, Gainesville, FL 32611-7048. This department explores rich mathematical ideas by revisiting their origins and early investigations found in the history of mathematics. Send submissions to this department by accessing* **mtms.msubmit.net**.

dar changed. The new revolutionary calendar was composed of twelve months of thirty days each and a short month of five days. The months themselves were also renamed. The late summer month, for example, was named *Thermidore* (the heat month), followed by *Fructidore* (the fruit month), and *Vendemiaire* (the vintage month). The days of the week were renamed in numeric terms according to their position: primidi, duodi, tridi, and so forth. Some citizens did not eagerly accept the new names. As Bishop Gregoire approached the guillotine, he scolded the revolutionaries with this statement: "Sunday has existed before you and it will survive you."

In the spirit of change, Count Talleyrand made a proposal in April 1791. He was a diplomat who had served the royal regime, the revolutionaries, Napoleon, and the returning royal family, in succession. He proposed to the National Assembly that the French Academy of Sciences devise a new measurement system that had logical, scientific backing, unlike the irregular development of the measurements used across France and Europe. On August 19, 1791, the formal proposal for the new system was presented to King Louis XVI, one day before his futile attempt to escape from France. Although he gave his enthusiastic approval, it is hard to know if he paid much attention, with his escape attempt only hours away.

In any event, the National Assembly soon directed the French Academy of Sciences to "deduce an invariable standard for all the measures and all the weights" (Johnson 1994). The new system of measures would be derived from the measurement of a line of longitude. Mathematicians, accountants, and scientists were assembled for the task, including scientist Antoine Lavoisier and mathematicians Adrien Legendre and Marie Condorcet. The group decided to use the basic unit

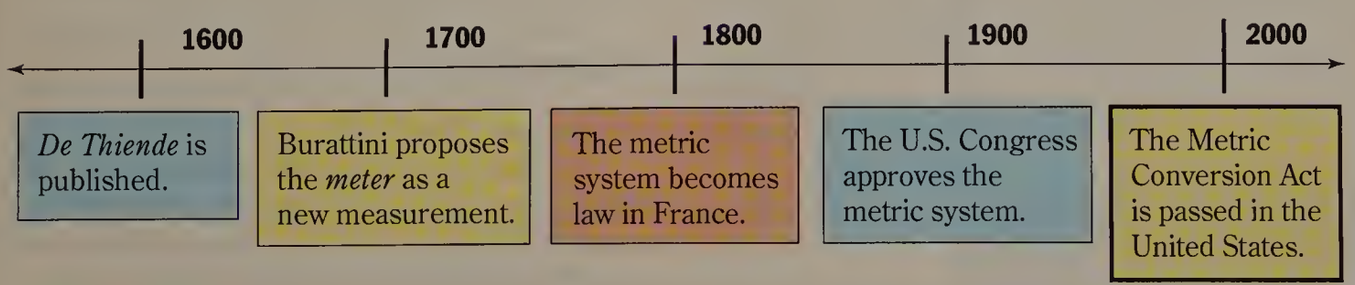
$$\frac{1}{10{,}000{,}000}$$

of the meridian length from the North Pole to the equator through Paris. This unit of length was called a "meter," derived from the Greek word *metron*, meaning "to measure." The meridian's length was to be determined by finding the length of a tenth of the meridian (or quadrant) of the distance from Dunkirk, on the English Channel, to Barcelona, Spain. This task, which would have been difficult under ideal conditions, took six years to complete because of frequent wars and changes in government and personnel.

The Provisionary Weights and Measures Committee appointed two astronomers, Jean Baptiste Delambre and Pierre Méchain, to survey the distance from Dunkirk to Barcelona using triangulation. From the start, the task was difficult and dangerous. Méchain was arrested while trying to leave Paris to begin work because his surveying instruments raised concern among the Revolutionary Guard. They feared Méchain was part of a plot to subvert the revolution and that the instruments he carried would be used to restore the monarchy. Fortunately, cooler heads prevailed, and Méchain and his assistants were freed from prison and a date with the guillotine. On another occasion, the surveyors were arrested because they were using white cloth as survey markers. White represented the royal family and using this cloth identified the surveyors as Royalists. The survey crew avoided the guillotine when they designed new sighting cloth containing the politically correct stripes of blue and red mixed with white. By 1793, the Revolution had claimed both Lavoisier and Condorcet as its victims. To avoid a similar fate, Delambre returned to Paris and turned in all his surveying instruments.

After laying out more than one hundred triangles (the triangle was used as a surveying unit of measure), the surveying team completed its task. In 1799, the metric system became law, eliminating the morass of weights and measures that had developed in France over the centuries. The metric system was presented to the French government, along with a platinum rod marking the meter and a platinum bar weighing a kilogram. The metric system, the official measurement system of France, took on this motto: "For all the people, for all time." The platinum rod and kilogram bar were stored in sterile conditions in the Pavillon de Berteuil, in Sèvres, outside Paris.

Old ways die hard, however, and in 1811 Pierre LaPlace (1749–1827) proposed calling the system Napoleonic Measures so as to insure its use and the Emperor's support. This new system did not win over the


1600
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De Thiende is published.
Burattini proposes the *meter* as a new measurement.
The metric system becomes law in France.
The U.S. Congress approves the metric system.
The Metric Conversion Act is passed in the United States.

French citizenry. In 1812, there was a brief return to metricized traditional units, which proved to be even more confusing than ever. In 1840, France returned to the metric system, never to depart again. Other European nations had adopted the system during the Napoleonic period as they fell under the influence of Napoleon's troops. Although Russia and England were holdouts, both came into the metric fold during the twentieth century. The United States remains the lone holdout among the world's industrialized nations.

Why didn't the United States adopt the metric system? As early as 1790, Thomas Jefferson had proposed some sort of metric measuring system. Some sources blame pirates who roamed the Atlantic Ocean 200 years ago. As the story goes, the French National Assembly sent a representative bearing a kilogram and a meter to the United States with news of that country's new measurement system. Pirates sank the representative's ship and enslaved all on board. Word of the new system did reach these shores, but along with the system came news of the excesses of the French Revolution, which poisoned many minds against anything metric.

Congress authorized the use of the metric system in national and local commerce with the passage of the Metric Act of 1866. In 1875, the United States and twenty other countries signed the Treaty of the Meter, which established the International Institute of Weights and Measures in Sèvres, France. Neither action furthered the use of the metric system in the United States. The Metric Conversion Act, passed in 1975, was "to coordinate and plan the integration of the metric system in the United States." Seven years later, the Metric Board was dissolved because of a lack of funding, interest, and success. In 1988, the Omnibus Trade and Competitiveness Act finally made some progress; four years later the Act mandated that any commercial contracts with the federal government must stipulate metric measures. As a result, the metric system entered the world of commerce, but the standard system of feet, pints, and pounds remains the entrenched measurement system of daily life in the United States.

There have been few changes in the metric system since it was first developed over 200 years ago. It continues to serve modern society as well as it served Napoleonic Europe. There have been advances in the standard meter, however. The original platinum rod with marks that designated the length of a meter is now simply an item of curiosity in a French museum, because it is inaccurate. Because Earth is not a perfect sphere, the original measurements by Méchain were in error. The actual length of

$$\frac{1}{10{,}000{,}000}$$

of the distance from the Equator to the North Pole is 0.2 millimeter longer than the meter inscribed on that platinum rod. Since 1983, a meter has been considered the distance that light travels in a vacuum in

$$\frac{1}{299{,}792{,}458}$$

of a second. (In a similar fashion, the second is now defined as the time it takes a cesium-133 atom to vibrate 9,192,631,770 times.) The kilogram remains the same as the platinum bar that was submitted to the French government 200 years ago. Le Bureau International des Poids et Mesures in Sèvres still houses the original kilogram, a cylinder of platinum in a bell-jar vacuum, which in turn is housed in another bell-jar vacuum.

Despite attempts to keep the original kilogram bar free of pollution and any other atmospheric effects by sealing it in multiple jars, it keeps gaining weight. In 1989, the last time the original kilogram, affectionately known by metric aficionados as Le Grand K, was removed for weighing and cleaning, scientists found that it had changed mass. It had gained, on average, about 20 billionths of a gram per year. Although this is roughly the weight of a pepper grain, it meant that the kilogram was not uniform.

The last vestige of the old way of measuring needs to be replaced with a standard that will never change. The problem for scientists is how to define a kilogram's weight in terms of modern science. One school of thought suggests defining the kilogram as a specific number of silicon atoms. However, silicon atoms have irregular atomic structures, never mind the sheer number of atoms needed for the definition. The number of gold atoms that weigh a kilogram is represented by twenty-five digits. Another more promising approach involves creating an electromagnetic force that exactly balances the force of gravity on a kilogram. Scientists at the National Institute of Standards and Technology in Gaithersburg, Maryland, have built a pan balance that is nearly two stories tall. Whatever combination of voltage and current balances a kilogram on this highly delicate balance would become the new definition of the kilogram. So far, researchers have spent more than twenty years making adjustments and readjustments to find the elusive numbers. Thus far, Le Grand K has resisted attempts to render it a mere number. Le Grand K lives, but for how long?

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SALLY ROBERTS AND
CARLA TAYEH

It's the Thought That Counts:

Reflecting on Problem Solving

WE STILL TALK ABOUT THE TIME WE were at the airport waiting to fly to a conference, both of us carrying formidable stacks of our students' metacognitive journals to read on the plane. As we looked at each other's papers and thought about the work ahead, we could not help but ask ourselves why we gave this assignment. As we began to read and share excerpts from students' journals, we did not need to ask why; we knew.

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For more than a decade, we have taught several similar courses for teacher candidates seeking elementary and middle school certification and have always been committed to teaching through problem solving. However, problem solving as the central theme in reformed mathematics curriculum convinced us that our students would benefit if we focused more explicitly on problem solving. Thus began a great adventure in collaboration and sharing. Having envisioned an experience that would immerse our students in the problem-solving process, the materials we developed engage students in solving nonroutine problems that require applying a variety of strategies (draw a picture, make systematic lists, look for a pattern, work backward, etc). We have found that students who genuinely enjoy problem solving *and* those who want to overcome their fear of the dreaded phrase "word problem" benefit from the experience. In addition to solving problems, students are given opportunities to monitor and reflect on their own thinking and performance as a mathematical problem solver. Pedagogical issues, including techniques for motivating students to solve problems and creating an atmosphere that supports risk taking, are routinely discussed in the context of the teacher candidates' own experiences. This relevant context blends pedagogy *and* content. We have shared our triumphs and challenges as well as our favorite resources, problems, and activities. The synergism in our collaboration reminds us that the whole is greater than the sum of the parts.

Background

RESEARCH SUPPORTS THE NOTION THAT WHAT one knows or believes about oneself as a learner and doer of mathematics can have a powerful effect on performance. It also highlights the critical influence that teachers' understanding of mathematics has on their pedagogical orientations and instructional decisions (Ball and Mosenthal 1990; Lappan and Even 1989; McDiarmid, Ball, and Anderson 1989). The *Professional Standards for Teaching Mathematics* (NCTM 1991) echoes this view:

> Teachers' comfort with, and confidence in, their own knowledge of mathematics affects both what they teach and how they teach it. Their conceptions of mathematics shape their choice of worthwhile mathematical tasks, the kinds of learning environments they create, and the discourse in their classrooms. (p. 132)

The ultimate goal is to create an environment in which students can grow as problem solvers. Students' *metacognitive skills* play a critical role in this process; these skills make students aware of both their own thinking and the thought processes of others as they explore problems in both whole-group and small-group settings. What is metacognition? Van de Walle (2004) describes it as the "process of consciously monitoring (being aware of how and why you are doing something) and regulating (choosing to do something or deciding to make changes) our own thought processes" (p. 54). Reys et al. (2004) describe the process as a "form of looking over your own shoulder—observing yourself as you work and thinking about what you are thinking" (p. 30).

Our motivation for focusing on metacognition and using metacognitive journals arose from three principles about metacognitive behavior. First, good problem solvers regularly monitor their thinking and are aware when they should rethink the problem or switch strategies. These metacognitive behaviors contribute to their success as problem solvers. Second, metacognitive behaviors can be learned (Garofalo 1987; Lester 1989). With this idea in mind, we searched for assignments and activities that would help students develop the skills to regularly reflect on and analyze their problem-solving process. Third, students who learn to monitor and regulate their own problem-solving behaviors do show improvement in their abilities (Schoenfeld 1987). When students are aware of their thinking patterns as they solve problems and reflect on weaknesses in their approach, they can use this information to grow in their problem-solving ability.

Getting Started

WE WANTED TO DESIGN MEANINGFUL ACtivities and assessment strategies so that students could thoughtfully discuss and reflect on how they solved problems. One way of doing this is by students presenting homework problems to classmates. Student volunteers prepare a transparency at home that captures how they solved the problem, which gives them time to reflect on how they solved the problem and to prepare a thoughtful presentation. Students discuss their problem and also solicit additional solution strategies from classmates. Students are then encouraged to write a narrative in class that expands on their problem solution in one of two ways: (1) by elaborating on either the key step that led them to the correct solution or (2) by identifying exactly what hindered them from finding the correct solution. Student volunteers share their narrative with the class. In addition to generating examples for the class, it gives the teacher and other students the opportunity to ask more probing questions about their thinking, which causes the students to think more deeply about the problem-solving process and become aware of how others are thinking. This is the

first step in modeling the writing of a metacognitive analysis for the class.

The culminating project is a metacognitive journal (see **fig. 1**), which allows students to reflect on their experiences and growth as problem solvers. The work is ongoing and represents a deeper and more thoughtful understanding of their problem-solving experiences. Homework problems, in-class problem-solving activities, class discussions, and in-class writing activities serve as models and preparation for the final project.

Metacognitive Journal

Choose any five problems over the semester for your metacognitive journal. Problems can be selected from in-class activities or homework problems. You should include the following sections for each problem:

1. Statement of the problem
2. Solution strategy
3. Reflections
4. Rationale for problem selection

Statement of the Problem

Restate or identify the problem you have selected.

Solution Strategy

Describe the strategy (or strategies) that you used to solve the problem. In this section, you will present a more polished version of your original work. For each problem, show how you solved it in addition to including a narrative that takes the reader through the steps you went through in your mind to find a solution.

Reflections

The following questions are intended to guide your reflections. You do not need to answer each of them.

- What did you do that helped you understand the problem?
- Did you find any information you did not need?
- Did you think about your answer after you got it?
- Can anything that you did in this problem help you to solve other problems?
- Does this problem remind you of other problems that you have solved?
- How did you decide your answer was correct?
- Did you have an "aha" moment?
- Did you try something that did not work?
- Did you encounter a stumbling block?
- Did someone in class (or outside of class) say something that gave you insight into the problem?
- Can you write and solve an extension to this problem?

Rationale for Problem Selection

In this section, you will provide a rationale for selecting each problem.

Fig. 1 An assignment sheet for the metacognitive journal

A Problem to Solve

WE HAVE ASSIGNED PROBLEMS FROM A VARIETY of resources, including *Problem Solving Strategies: Crossing the River with Dogs* (Johnson and Herr 2001). This book contains a good collection of nonroutine problems and models the metacognitive process through examples of students' solutions in each chapter. We look for problems that stimulate discussion, require students to justify their answers, and are accessible for students at different stages in their development as problem solvers. The Ups and Downs of Shopping is one such problem (see **fig. 2**). On the surface, it appears to be straightforward, and students tend to think they can do the problem in their heads. In reality, it is an interesting problem that models the process of metacognitive analysis for students. Before reading on, try solving the problem yourself.

> Roberto is shopping in a large department store with many floors. He enters the store on the middle floor from a skyway, and he immediately goes to the credit department. After making sure his credit is good, he goes up three floors to the housewares department. Then he goes down five floors to the children's department. Then he goes up six floors to the TV department. Finally, he goes down ten floors to the main entrance of the store, which is on the first floor, and leaves to go to another store down the street. How many floors does the department store have?

Fig. 2 The Ups and Downs of Shopping problem (from Johnson and Herr 2001, p. 24)

After students have an opportunity to solve the problem individually, they share their answers with the class. Some popular answers include 10, 12, and 13. At this point, we allow students time to discuss and justify their answers with their peers. They are always steadfast in their commitment to their answers and are anxious to convince classmates that they are right. Although some disagreement is found among the students, we reconvene the class as a whole and solve the problem step by step using a diagram. After the class agrees that the answer is 13, we focus on solution strategies and why this problem generates so much uncertainty when at first glance it appeared to be so easy. Using this problem early on in the course engages students in a lively discussion that begins their journey into reflective thinking.

Students take a second look

When students have an opportunity to revisit problems in their metacognitive journals, several always

select The Ups and Downs of Shopping. One student wrote the following:

> For this problem, I broke all of the information down and drew a diagram. First, I drew a tall rectangle to represent the store, but I left the lines on the top and the bottom off so I could make it *taller or shorter*. I knew that Roberto started on the middle floor, so I drew a line in the middle, this also told me that however many floors I added on top must also be added on the bottom, and vice versa. After following Roberto around the store, I discovered that there are 13 floors in the department store. I chose this problem because it was kind of fun.

The student's comment that it was fun was particularly insightful in light of having identified herself as a tentative problem solver. Her reflections suggest that the problem helped build her confidence, and it is one of numerous examples where students equate *success* with *fun* in their journals.

Problem selection

Before students turn in their journals, they anxiously compare notes to see if they have selected any of the same problems. The students' rationales for selecting problems represent a range of emotions: "I selected the problem because I was successful [or not successful] and wanted 'a second chance' to revisit the problem"; "the problem made me feel empowered"; "humbled"; "fascinated"; or "frustrated." As we read through the journals, we discovered that some themes reoccur, which inform us about our students as well as our teaching.

Lessons learned by problem solvers

Since one of our principles is that good problem solvers monitor their thinking as they solve problems, we examine our students' journals for evidence of this behavior. After reading their journals, it is clear that students are critically examining their thought processes and growth as problem solvers. The following lessons represent aspects of this process. We have added italics to the students' journal entries for emphasis.

Lesson 1: Insights into one's own strengths and weaknesses. The first step in becoming a better problem solver is to identify both personal strengths and weaknesses. Students often focus their discussions in their journals on what they have learned about their strengths and their limitations as problem solvers. One student recognized that she needed to work on her patience and persistence as a problem solver:

> I realized that *I was good at what I knew*. It turns out that there is a lot more to problem solving than I had thought. I've realized that I have difficulty continuing with problems that I'm getting nowhere with.

Another student focused on her flexibility as a problem solver:

> I was unable to *think outside the box*. I felt that I was easily stumped. *I lacked the skills to be flexible*, partly because I lacked the knowledge about various strategies that could be used for problem solving.

Lesson 2: A road map for improving problem-solving skills. Being aware of strengths and weaknesses is only the beginning; committing to a plan for changing problem-solving habits so as to continue growth is the next step. This student reaffirmed her commitment to checking her answers:

> I started the problem out correctly. However, when I started going through the clues, I made one fateful mistake and it ruined everything. This problem illustrates my biggest flaw as a problem solver, I don't re-check my answers.

Another student's journal entry reflected her vow of becoming a more flexible problem solver and her commitment to using a variety of strategies:

> I usually try only one problem solving strategy. When that doesn't work then *I start to doubt myself*. I'm finding that I usually have to use more than one strategy to get my answer.

Many students acknowledged that good problems take time to solve and voiced their commitment to becoming more persistent problem solvers. One student's remarks captured this newfound understanding and commitment:

> I have a tendency to read through problems very quickly and want to find the end result quickly. I have found that it is necessary to SLOW DOWN and think things through.

Lesson 3: Awareness of one's problem-solving disposition. Mathematical dispositions are guided by beliefs, attitudes, and intuitions. They also influence how an individual approaches problem-solving tasks. If students are to develop a positive disposition to do mathematics, it is essential that their teacher communicate an excitement for, and confidence in, doing mathematics (NCTM 1991). Future teachers of mathematics must reflect on their own dispositions about problem solving and confront and overcome their own limitations so that they do not impose them on students. One teacher candidate noted her own limitations by writing that "I got frustrated. I should have left the

problem for at least a day and gone back to it with a clean sheet of paper, not the one I had previously used. *This way any fresh ideas won't be smothered.*" Initially, many of the teacher candidates look for an algebraic expression that will quickly unlock the solution to a problem. Another teacher candidate noted in her journal that she valued her ability to offer alternative solution strategies and hoped that her peers were able to learn from her solutions: "I feel like I have contributed a less algebraic form of problem solving. In most instances members in my group solved the problem algebraically, while I could share ways in which our future students may solve the problems."

Lessons learned by teachers

The students are not the only ones who learn from this assignment. We also gain insights from listening to our students' views and perspectives of our classrooms. These lessons help us to reflect on our own practice and reshape both the assignments and the instructional delivery.

When students write about and reflect on their own thinking, it makes a significant impact on their ability to solve problems now and in the future

Lesson 1: Understanding how students perceive their own growth as problem solvers. We are committed to the belief that metacognitive behaviors can be learned and that students who learn to monitor and regulate their own problem-solving behaviors do show improvement. As students reflect on their growth as problem solvers, they will typically focus on their willingness to persevere when solving problems. The journals seem to provide concrete evidence that although problem solving takes time and can often feel uncomfortable, students are more often successful when they do not give up. For example, students wrote:

> I am definitely a better problem solver. I'm more creative and original, and I don't have as many mind sets. I also *work harder and longer at problems*, instead of giving up quickly.

> I've learned *to let myself be frustrated for a while* as long as I came back to the problem later with a clear mind.

Lesson 2: Awareness of students' attitudes and dispositions about problem solving. Teacher candidates' dispositions toward problem solving will impact the climate they create in their own classrooms, and their journals are one tool for assessing attitudes and dispositions. Our hope is that teacher candidates will enter their own classrooms armed with confidence and excitement to engage their students in problem solving. Comments from students provided a snapshot of their dispositions about problem solving:

> At first glance this problem seems quite complicated, which is also *why it seems so enticing.*

> I just kept on plugging and chugging until I solved the answer correctly. I realized I really like problems like this. *I find it really fun to solve hard problems.*

Lesson 3: Enrich your awareness of how students think about mathematics. Beliefs and intuitions about mathematics also shape how we do and teach mathematics. Many students hold untrue beliefs about mathematics, such as that *all* mathematics problems have one and only one right answer; that there is only one way to solve a problem; that students are not expected to understand mathematics, just memorize facts; and that students should be able to solve a problem in ten minutes or less (Schoenfeld 1994). It is unfortunate that teachers perpetuate many of these misconceptions. Confronting these myths and misconceptions is the beginning of changing classrooms of the future. Journal entries indicated that teacher candidates are on their way to changing their beliefs:

> I have learned that math is not always about *clean-cut answers that pop out of formulas.*

> I am better at finding more than one alternative method to derive the same or different solutions. I found myself looking for optional solutions that would still be acceptable but would be the result of *making different assumptions from the problem statement.*

Conclusion

Although we have learned from teacher candidates, these same lessons can be learned in any classroom where problem solving is being taught. Encouraging students to write about their thinking can provide unique insight into the way students are thinking about the mathematics that they are investigating. In addition, when students write about and reflect on their own thinking, it makes a significant impact on their ability to solve problems now and in the future.

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Do You Want to Add *Your* Two Cents?

You may have opinions about articles or departments mentioned in this issue. Perhaps you have a classroom activity to share that is related to one of them. If so, please share them with other teachers by sending an e-mail to **mtms@nctm.org**, writing "Readers Write" in the message line.

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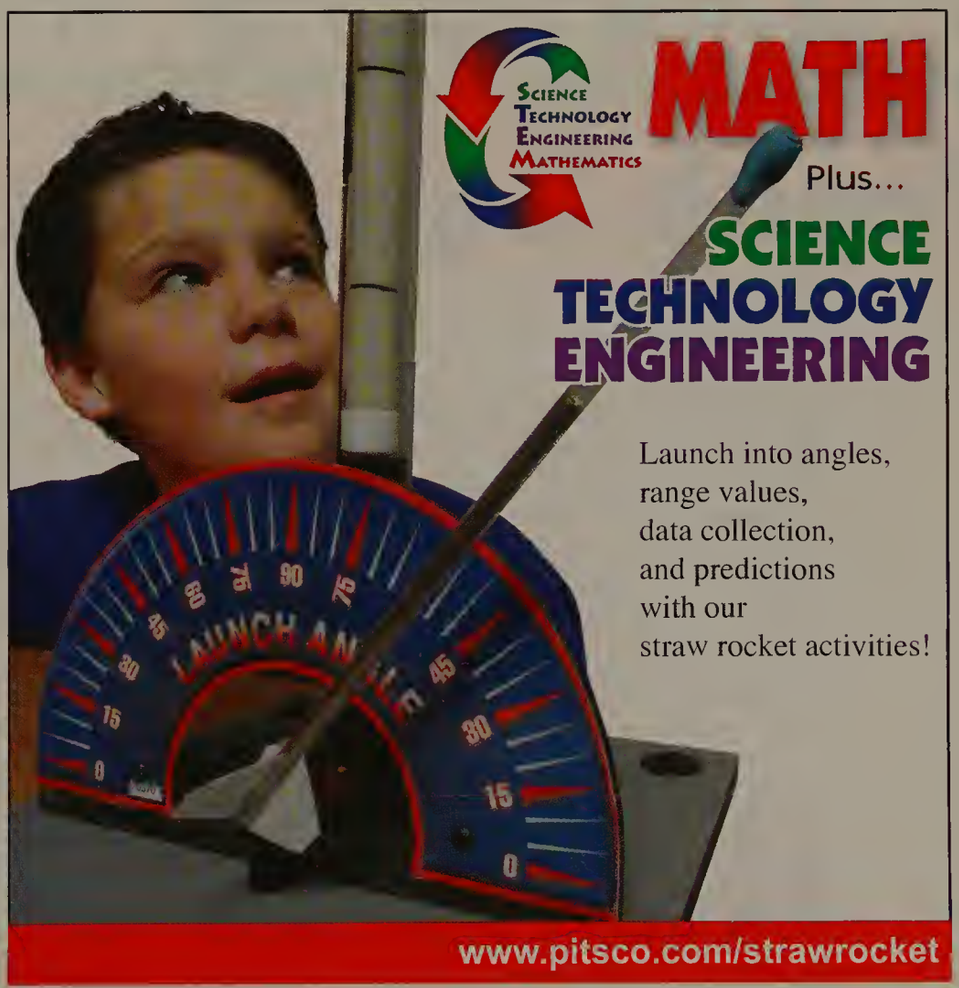

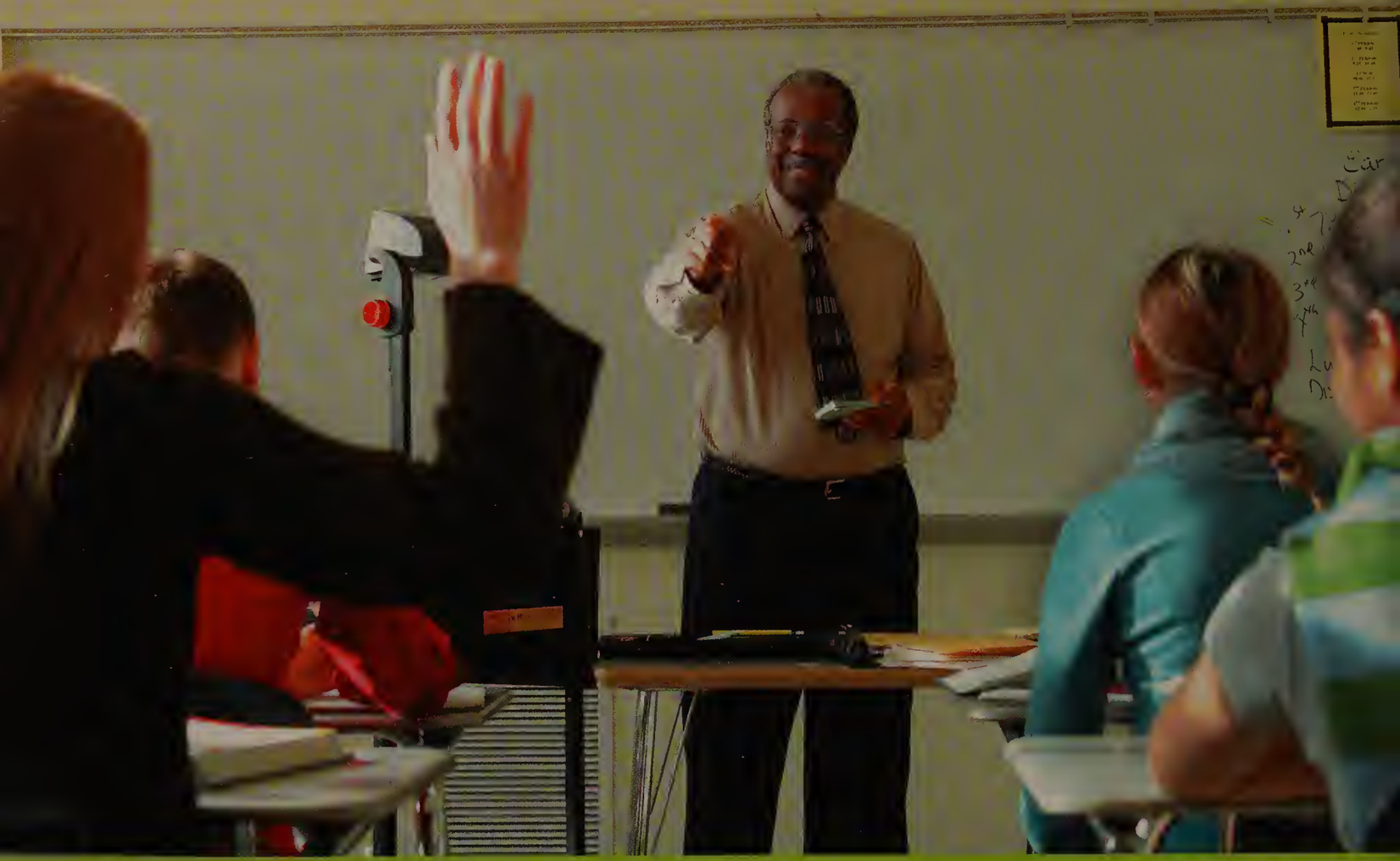

Using Error Analysis to Teach Equation Solving

KATHY HAWES

IN MANY MATHEMATICS CLASSROOMS, students are responsible for correcting their own homework. Placing a red X next to an incorrect answer is often the extent of homework checking because students often lack the ability to find and correct mistakes in their own work. Without this skill, students lose a valuable opportunity to discuss homework in a meaningful way. When classroom activities and discussion focus on analyzing errors, students realize that finding a numeric answer is an insufficient goal and become more willing to show and explain their work. Homework discussions begin to focus on "How did you get that?" rather than being limited to "Is my answer right?"

According to the NCTM's *Principles and Standards for School Mathematics*, "Students who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically" (NCTM 2000, p. 60). This article presents several Standards-based activities and problems that use error analysis to improve students' ability to read and discuss mathematics work. In the first two activities, students are required to explain their thinking and listen carefully to the thinking of other students.

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Written on the Board	Student Explanation
$2(x + 1) = x - 4(x - 2)$	Teacher presents problem.
$2x + 2 = x - 4(x - 2)$	Student 1 uses the distributive property.
$2x + 2 = x - 4x - 8$	Student 2 uses the distributive property incorrectly. This step is later circled and identified as a common error.
$2x + 2 = x + -4(x + -2)$	Amanda writes her idea on the board.
$2x + 2 = x + -4x + 8$	Student 2 corrects the error after discussion.
$2x + 2 = -3x + 8$	Student 3 combines like terms.
$3x + 2x + 2 = 3x + -3x + 8$ $5x + 2 = 8$	Student 4 adds $3x$ to both sides to get all the variable terms on one side.
$5x + 2 - 2 = 8 - 2$ $5x = 6$	Student 5 writes an alternate solution from his homework paper, explaining that he subtracted 2. $2x + 2 - 2 = -3x + 8 - 2$ $2x = -3x + 6$ Then he subtracts 2 from student 4's solution.
$x = 6/5$	Student 6 divides both sides by 5 because division "undoes" multiplication.

Fig. 1 A typical problem solved during "pass the pen"

Through the third activity, in-class journal writing and take-home writing projects, students develop an understanding of common errors and improve their ability to communicate effectively with others when correcting homework. Although all the examples come from a unit on linear equation solving, the ideas and activities presented in this article have been used successfully in other algebra units and in other courses.

Activity 1: Pass the Pen

ACCORDING TO THE NCTM'S STANDARDS, "FOR some students, participation in class discussions is a challenge" (NCTM 2000, p. 61). The "pass the pen" activity is structured so that class discussion involves every student. The teacher writes a multistep mathematics problem on the whiteboard and calls on one student to come up and complete the first step. The teacher gives the student a whiteboard marker, or pen, to write his or her work. The student explains not only how to complete the step but also the reasoning behind it. The student holding the pen then calls on a second student to complete the next step, and "passes the pen." After completing a step, each student passes the pen to another until the problem is finished. When a question arises, the person holding the pen must answer the question, call on another student for help, or pass the pen to a different student. Typically during this activity, one or more errors will be made by students, as in the following scenario referring to **figure 1**.

Student 2: I used the distributive property, like in the first step. [*Hands go up all over the room. Student 2 is alerted by this action that she has made a mistake and looks to the teacher.*] Isn't that right?

Teacher: If you have a question, why don't you call on someone to help you? [*Student 2 looks around the room and chooses Mario.*]

Mario: You forgot about the subtraction sign in front of the 4.

Student 2: [*Obviously still confused.*] I don't get it. That's why it's subtracting $4x$. I multiplied the 4 and the 2 to get 8. [*She calls on Amanda for more assistance.*]

Amanda: If you write subtraction as adding the opposite, the problem is –4 times –2 and that's +8. You got –8 because you didn't look at both negatives.

Teacher: Amanda, will you write your idea on the board so that everyone can better understand what you are saying? [*Student 2 hands Amanda the pen, and she writes her idea on the board. When Amanda is finished, she returns the pen to student 2.*]

Student 2: Oh, I get it. A negative times a negative is a positive. You have to write the subtraction as an addition problem to see both negatives. [*She corrects her work and passes the pen to a new student.*]

Solving a problem in this manner will clearly take more time than a teacher-presented solution given during a lecture. The extra time is necessary, because students require time to think between each step and formulate their questions. During a lecture, many middle-level students have difficulty simultaneously

copying notes and processing the ideas. As an additional benefit, the teacher has an informal opportunity to assess what concepts are still giving students difficulty, as in the dialogue below when student 5 is uncertain how to proceed.

Student 5: That's not what I got [*referring to student 4's work*]. I don't know what to do next!

Teacher: Why don't you write what you did on the board and call on someone to help you? [*Student 5 copies work from his homework paper.*]

Student 5: [*Looking for hands.*] I got $2x = -3x + 6$; does that mean I'm wrong? [*Student 5 calls on Tran.*]

Tran: I did what you did, and I think it's right because I just subtracted 2 from both sides then added $3x$. Student 3 has already added the $3x$, so you can subtract 2 from both sides now, and it will still work. [*Student 5 subtracts 2 from both sides.*]

Teacher: [*To student 5.*] Does that agree with the answer you got on your homework?

Student 5: Yes. [*Student pauses and thinks.*] Can you always do the steps in any order?

The discussion continues by students passing the pen to other students. When disagreements or different solutions occur during the discussion, it is helpful to write alternative solutions on the board in addition to the original solution. Students will benefit from the opportunity to compare solutions and judge whether the differing solutions are both correct. If both solutions are correct, students can form their own opinion about which solution is easier. If a solution is incorrect, it is helpful to add notes to clarify the error so that students can avoid a similar mistake in future work.

A key element of "pass the pen" is that everyone must participate. No one can go to the board a second time until everyone has held the pen at least once. Students who usually do not present solutions at the board know that they will have to take a turn and look for an opportunity to participate. This activity is also effective because the person who holds the pen is always in charge of the discussion, not the teacher. Thus, although everyone must take a turn, he or she can always call on another student for help or pass the pen if feeling uncomfortable. The role of the teacher during "pass the pen" is to enforce the participation rules and ask questions that will take the discussion deeper: Why did you do that? Is that the only possible next step? Will that always work? How is that different from the last problem?

Activity 2: Find an Expert

IN THIS ACTIVITY, STUDENTS TAKE TURNS BEING the expert and the questioner. It is a useful way for students to correct their own work on a quiz or homework assignment. Students love the opportunity to move around the room and work with different people. They also appreciate playing different roles in the activity. The teacher can assign roles for particular problems or allow students to select their own partners.

Students are instructed to write in a binder the correct solutions to the problems they missed by consulting a different expert for each missed problem. An expert, someone who solved the problem correctly, is instructed not to show his or her own paper to anyone but to find mistakes and answer questions. The questioner, the student who missed the problem, is to ask enough questions to be able to write the correct solution on the binder paper.

"It is difficult for students to learn to consider, evaluate, and build on the thinking of others, especially when their peers are still developing their own mathematical understandings" (NCTM 2000, p. 63). The rules of this activity discourage copying and encourage discussion of students' thinking. Because the expert cannot write on the binder paper, he or she must use words to explain his or her thinking. It is useful to model good questions before the activity begins. For example, "Why did you divide by 3?" is a more thoughtful question than "What should I write next?" During the activity, the teacher monitors the

Common Error	Reason for Confusion (False Analogy)
$2x + 3x = 5x^2$	Students confuse $2x + 3x = 5x$, where only the coefficients are added, with $2x \cdot 3x = 6x^2$, where the coefficients as well as the variable are multiplied.
$-3^2 = 9$	Students remember that the square of a negative is positive but forget the importance of parentheses. $(-3)^2 = +9$ and $-3^2 = -9$
$6x - (x - 3) = 6x - x - 3$	Students fail to understand that subtracting quantities is different from adding quantities. Another problem can look similar to students. $6x + (3x - 3) = 6x + 3x - 3$
$-\|-2\| = 2$	Students find it difficult to distinguish between absolute value and parentheses. $-(-2) = +2$ but $-\|-2\| = -2$

Fig. 2 Various ideas for writing prompts

discussions in the classroom. A conversation where an expert explains a solution incorrectly provides an excellent opportunity for the teacher to ask clarifying questions of his or her own. Sometimes a pair of students become stumped and are unable to figure out the mistake in the questioner's method. This situation also provides an excellent problem for further class discussion or journal writing.

Activity 3: Journal Writing

STUDENTS ARE PREPARED TO DESCRIBE THEIR thinking in journals after discussing problem solutions and working collaboratively to find errors. It is helpful to have students write in journals on a regular basis and organize journal entries around common errors (see **fig. 2**). A careful reading of student responses can give insight into the current depth of understanding for each student. When asked to find the mistake in the following journal entry, three students demonstrate different levels of understanding (see **fig. 3**).

Gabrielle attempts to prove the simplification is incorrect by evaluating the expression. Although she is clearly aware that the order of operations is required, her work is unclear and she is unable to find the mistake or the correct simplification. Prabdheep is also unable to correctly simplify the expression, although his explanation of the distributive property is clear. He correctly explains that the +6 should be −6, but he believes the student should also replace $1 - x$ with $-1x$. Prabhdeep forgets to add the constant term of 1 in the first quantity of $(2x + 1)$ and introduces a new error. Only Briannon simplifies the expression correctly by equating the subtracting of quantities and the distributing of a negative.

Activity 4: Exams and Take-Home Projects

"TO ENSURE DEEP, HIGH-QUALITY LEARNING FOR all students, assessment and instruction must be integrated" (NCTM 2000, p. 23). It is essential that

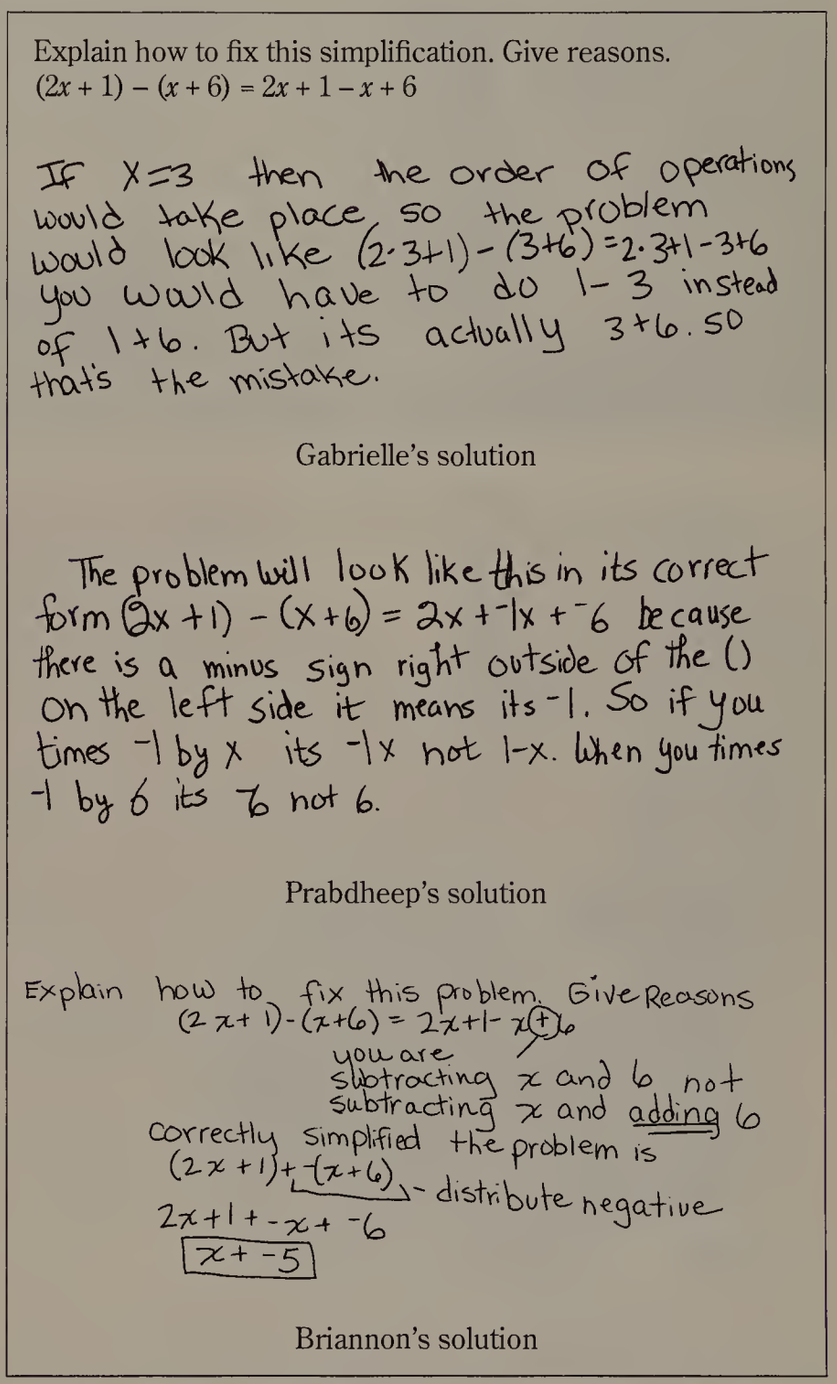
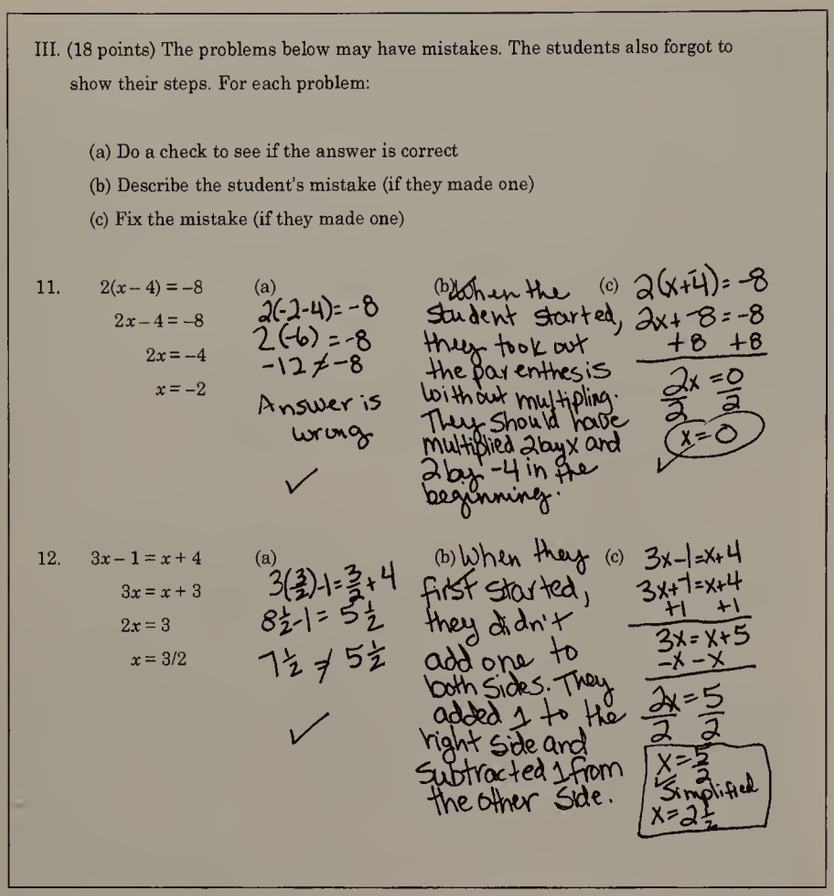

Explain how to fix this simplification. Give reasons.
(2x + 1) − (x + 6) = 2x + 1 − x + 6
If X=3 then the order of operations would take place, so the problem would look like (2·3+1)-(3+6)=2·3+1-3+6 you would have to do 1-3 instead of 1+6. But its actually 3+6. so that's the mistake.
Gabrielle's solution
The problem will look like this in its correct form (2x+1)-(x+6)=2x+-1x+-6 because there is a minus sign right outside of the () On the left side it means its -1. So if you times -1 by x its -1x not 1-x. When you times -1 by 6 its -6 not 6.
Prabdheep's solution
Explain how to fix this problem. Give Reasons
(2x+1)-(x+6)=2x+1-x(+)6
you are subtracting x and 6 not subtracting x and adding 6
correctly simplified the problem is
(2x+1)+-(x+6) - distribute negative
2x+1+-x+-6
x+-5
Briannon's solution

Fig. 3 A sample journal problem, listed at top, is explained three ways by three students.



III. (18 points) The problems below may have mistakes. The students also forgot to show their steps. For each problem:

(a) Do a check to see if the answer is correct

(b) Describe the student's mistake (if they made one)

(c) Fix the mistake (if they made one)

11. $2(x-4) = -8$

$2x - 4 = -8$

$2x = -4$

$x = -2$

(a) $2(-2-4) = -8$, $2(-6) = -8$, $-12 \neq -8$. Answer is wrong. ✓

(b) When the student started, they took out the parenthesis without multiplying. They should have multiplied 2 by x and 2 by -4 in the beginning.

(c) $2(x+4) = -8$, $2x + 8 = -8$, $-8 \quad -8$, $\frac{2x}{2} = \frac{0}{2}$, $x = 0$ ✓

12. $3x - 1 = x + 4$

$3x = x + 3$

$2x = 3$

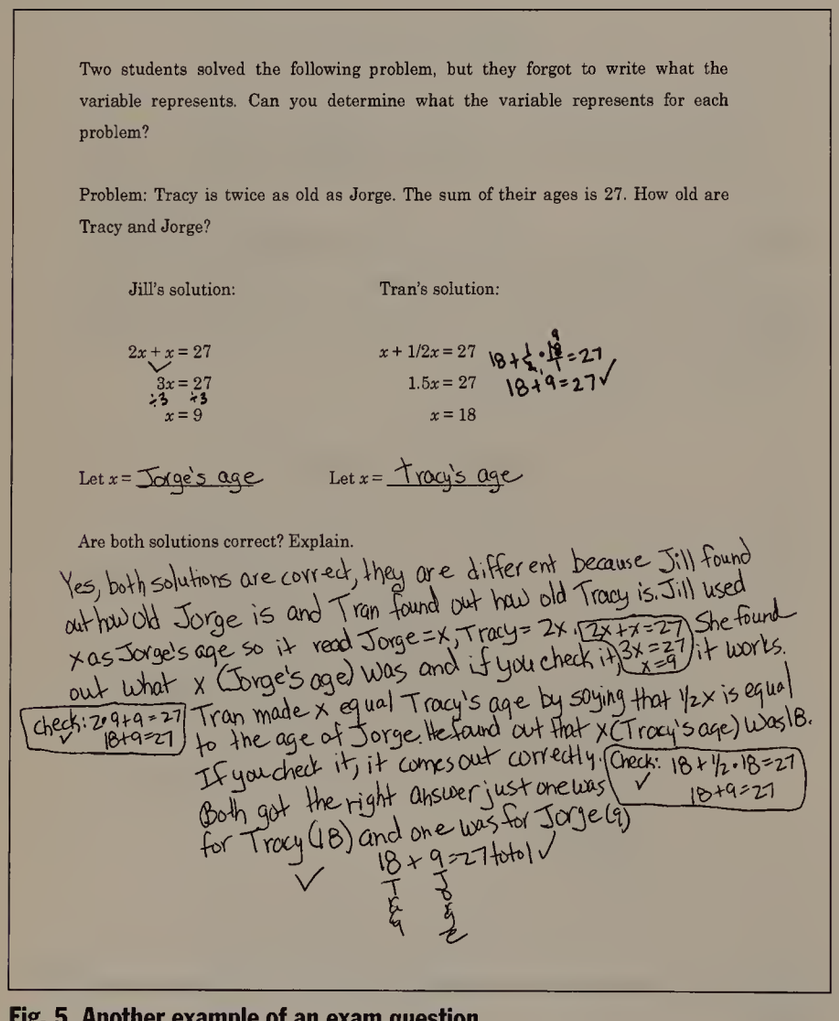
$x = 3/2$

(a) $3(\frac{3}{2}) - 1 = \frac{3}{2} + 4$, $8\frac{1}{2} - 1 = 5\frac{1}{2}$, $7\frac{1}{2} \neq 5\frac{1}{2}$ ✓

(b) When they first started, they didn't add one to both sides. They added 1 to the right side and subtracted 1 from the other side.

(c) $3x - 1 = x + 4$, $3x + 1 = x + 4$, $+1 \quad +1$, $3x = x + 5$, $-x \quad -x$, $\frac{2x}{2} = \frac{5}{2}$, $x = \frac{5}{2}$, Simplified $x = 2\frac{1}{2}$

Fig. 4 Sample exam questions



Two students solved the following problem, but they forgot to write what the variable represents. Can you determine what the variable represents for each problem?

Problem: Tracy is twice as old as Jorge. The sum of their ages is 27. How old are Tracy and Jorge?

Jill's solution:

$2x + x = 27$ ✓

$3x = 27$

$\div 3 \quad \div 3$

$x = 9$

Let $x =$ Jorge's age

Tran's solution:

$x + 1/2x = 27$

$1.5x = 27$

$x = 18$

$18 + \frac{1}{2} \cdot 18 = 27$, $18 + 9 = 27$ ✓

Let $x =$ Tracy's age

Are both solutions correct? Explain.

Yes, both solutions are correct, they are different because Jill found out how old Jorge is and Tran found out how old Tracy is. Jill used x as Jorge's age so it read Jorge = x, Tracy = 2x. $2x + x = 27$, $3x = 27$, $x = 9$. She found out what x (Jorge's age) was and if you check it it works. Check: $2 \cdot 9 + 9 = 27$, $18 + 9 = 27$ ✓ Tran made x equal Tracy's age by saying that 1/2x is equal to the age of Jorge. He found out that x (Tracy's age) was 18. If you check it, it comes out correctly. Check: $18 + \frac{1}{2} \cdot 18 = 27$, $18 + 9 = 27$ ✓ Both got the right answer just one was for Tracy (18) and one was for Jorge (9). $18 + 9 = 27$ total ✓

Fig. 5 Another example of an exam question

students have the opportunity to demonstrate their ability for error analysis on tests and projects. This format provides an effective measure of understanding, because students must explain their thinking. Test questions should be short and focus on a single concept. The sample questions in **figure 4** give students the opportunity to demonstrate their understanding of the distributive property in the first question and the addition property of quality on the second problem (see **fig. 4**). Note that there are student errors in problem 12; the simple error is easily found using this format. A third problem assesses students' ability to set up word problems (see **fig. 5**).

Including error analysis as part of a take-home test or project presents students with the opportunity to demonstrate their knowledge on more complicated multistep problems. A typical take-home assessment requires students to identify two or three errors in a single solution, write a correct solution, and justify their reasoning for each step. At home, students do not work under the same time pressure as in-class tests and have the opportunity to express their mathematical thinking more thoughtfully.

Conclusion

ALLOWING OPPORTUNITIES TO DISCUSS AND EXplain common errors can help students become more accurate at solving linear equations and working cooperatively to correct their homework. Students are most effective when they have the opportunity to explain their thinking daily in classroom discussions as well as in smaller collaborative groups. Activities like "pass the pen" and "find an expert" should be used once or twice during a unit to foster active student discussion and participation. As students become better at error analysis, they should be given the opportunity to work alone and explain their ideas in journal writings several times a week and on graded tests and take-home projects at the end of the unit. Questions requiring error analysis are effective tools to assess students' understanding of linear equation solving because error analysis requires a deeper understanding of equation solving than simply "doing the problem."

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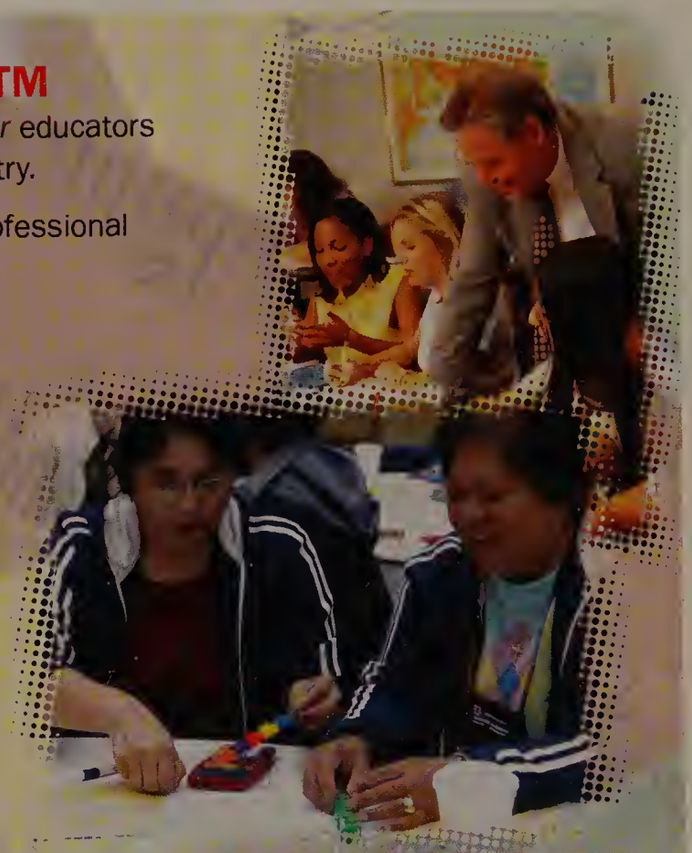

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SUSAN B. TABER

Using *Alice in Wonderland* to Teach Multiplication of Fractions

ALICE'S ADVENTURES IN WONDERLAND BY Lewis Carroll is a captivating story that appeals to students in the middle grades. It also provides an opportunity to help students develop an understanding of some complex mathematical content, particularly the multiplication of fractions, which is introduced in the middle grades.

First published in 1865, *Alice's Adventures in Wonderland* began as a story told by Charles Lutwidge Dodgson, an Oxford don and lecturer in mathematics, to the three Liddell sisters, ages 8 to 13, during an afternoon's boat trip in 1862. The story might have vanished into the summer's air but for Alice Liddell, age 10, who begged Dodgson to write it down. The first version, *Alice's Adventures Underground*, was handwritten and illustrated by Dodgson and presented to Alice Liddell as a Christmas gift in 1864. For publication, Dodgson revised and expanded the story, which was then illustrated by John Tenniel. He also changed the name to *Alice's Adventures in Wonderland*, and the book was published as the work of Lewis Carroll (Cohen 1995).

Alice's Adventures in Wonderland can be used as the focus of an interdisciplinary language arts and mathematics unit. The story has a Lexile rating of 890 and is considered to be at an appropriate reading level for students in grades 4 through 8 (Scho-

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lastic 2004). (*A Lexile rating is a scale used to determine the reading level of a particular text.—Ed.*) The story is familiar to most children who have either read the book or who have seen the movie; that familiarity can support students' understanding of the text. The following list describes four crucial mathematics concepts related to multiplication of fractions and proportional reasoning that can be developed during an interdisciplinary unit:

1. Multiplication by a number less than 1 results in a product that is less than the original number, and multiplication by a number greater than 1 results in a product that is more than the original number
2. Multiplication by a very small number will result in a very small product but will not result in a product of 0
3. The distinction between additive/subtractive change and multiplicative change
4. The multiplicative relationships that are at the heart of proportionality and similarity

Multiplication by a number less than 1. The multiplication of fractions is a complex mathematics concept and is often difficult for students *and* adults. Studies have shown that even after learning to compute the products of fractions or decimals, most students have difficulty solving multiplication word problems involving fractions or decimals less than 1 (Graeber 1993; Greer 1992). For example, after two weeks of instruction on multiplication of fractions, the majority of students in a fifth-grade class solved problems like the following by dividing the whole number by the denominator and multiplying the result by the numerator.

> JoAnne's dog weighs 21 pounds on earth. On a smaller planet it would weigh 2/3 as much. How much would it weigh on the other planet?

On the other hand, the students used multiplication to solve a problem like this one:

> Dave's science report was 1/3 of a page long. Frank's history report was 12 times as long as Dave's report. How many pages long was Frank's report?

Even though the students knew how to correctly find the product of two fractions and of whole numbers and fractions, most of them did not think of the first of these two problems as being a multiplication problem even though it was placed immediately after the science-report problem on the page.

One reason for this is that the two factors in multiplicative situations play different roles. One factor is the operator or multiplier; it affects the change in the other factor, the quantity. Although people easily recognize problems that have whole-number multipliers as being multiplication situations, difficulties arise when the multipliers are fractions or decimals less than 1. Many students and adults think that these situations do not involve multiplication but rather division or subtraction because the resulting quantity is smaller than the quantity named in the problem (Taber 1999). If students become accustomed to solving problems like the first one (JoAnne's dog) by dividing and then multiplying, they may come to believe that problems that ask them to find a fraction of a quantity are solved by division. If the problem asked them to find 2/3 of 21 1/2 pounds or of 5/8 pound, however, the method of dividing by the denominator and multiplying by the numerator would be much more difficult to carry out.

Instruction on multiplication of fractions, therefore, needs to help students extend and transform their understanding of whole-number multiplication to include new kinds of problem situations. For students to understand that multiplication can solve problems with fraction multipliers requires them to extend and reconceptualize their understanding of multiplication to include multiplication by fraction multipliers.

In teaching a unit on multiplication of fractions to another class of fifth-grade students, I used the story of *Alice's Adventures in Wonderland* to help students understand that multiplication can shrink quantities as well as enlarge them. The students and I reviewed the various things that happened to Alice after she ate cake or drank potions from bottles. "Did you know," I asked them, "that Lewis Carroll was really a mathematician named Charles L. Dodgson? Let's see how we can show with mathematics the kinds of things that happened to Alice in the story. Let's suppose that Alice is 4 1/2 feet tall, that's 54 inches." I wrote "54 inches" on the board and drew a little bottle next to it. "Now we're going to have her drink from this little bottle that will make her 1/9 of her present size," I said as I wrote 1/9 on the bottle. "How tall will she be after she drinks?"

"Six inches," the students replied. I wrote "=6" on the board to the right of the bottle.

"That's pretty small," I said. "Suppose it was a 1/3 bottle she drank from instead?" I wrote "54" and drew a bottle labeled 1/3 next to it. "How tall would she be then?"

"Eighteen inches," Jamal said, and the rest of the class nodded in agreement. I recorded "=18 inches" on the board.

"Let's let her drink from a . . ." I began, but was interrupted by Donald, who said, "One-sixth bottle."

I wrote "54" and a bottle labeled 1/6 on the board.

"That's 9 inches," Tamika said. I wrote "= 9 inches" on the board.

"Now I want to see what would happen if she drank from a 5/6 bottle," I said, writing 54 followed by a bottle labeled 5/6.

"Oh, this is easy," said several students. "It's 45 inches."

I asked Ken to explain how he found 45 inches. "If 1/6 is 9 inches," he said, "then 5/6 would be 9 times 5 and that's 45."

I called the students' attention to the list we had made on the board (**fig. 1**). "This shows that we're changing Alice's height when she drinks from the bottles. What mathematical symbol should we put in here between Alice's height and the bottles?"

Alice's Height

Start	Bottle	Finish
54 inches	1/9	= 6 inches
54 inches	1/3	= 18 inches
54 inches	1/6	= 9 inches
54 inches	5/6	= 45 inches

Fig. 1 Changes in Alice's size

ILLUSTRATION BY JOHN TENNIEL, FROM *ALICE'S ADVENTURES IN WONDERLAND* BY LEWIS CARROLL

Fig. 2 Alice and the Caterpillar

"Division," Jamal said. "You're dividing 54 by 1/9."

Stephanie raised her hand. "It's multiplication. You're multiplying 54 by 1/9."

"How many think we're dividing?" I asked. About a third of the students raised their hands. "How many think we're multiplying?" Another third raised their hands. "How many are not sure? Let's take a small Alice and make her get bigger. Maybe that will help us figure out what's going on." We then pretended that Alice ate cakes labeled with various whole numbers and fractions greater than 1. Students computed her height after she ate each cake. We then discussed the appropriate operation symbol to put between the beginning height and either the cake or the bottle. Students agreed that the multiplication symbol would work in either situation: when Alice was shrinking as well as when she was growing.

Before beginning the lesson the next day, one student's question indicated to me that the students were beginning to understand that multiplication could indeed decrease a quantity. Carrie asked, "Why is it when we times 29 times 2/9 that the answer goes down?"

I wanted to be sure that I understood her question, so I asked one of my own: "Your question is that if we multiply 29 times 2/9, it's smaller than what?"

"Twenty-nine," Carrie replied. I asked if anyone in the class could think of a reason.

Nina quickly raised her hand, "If you times 29 times 1, you get 29, but 2/9 isn't a complete 1, so you get less than 29."

Alan contributed, "Twenty-nine 2/9 is like counting 2/9 [a total of] 29 times."

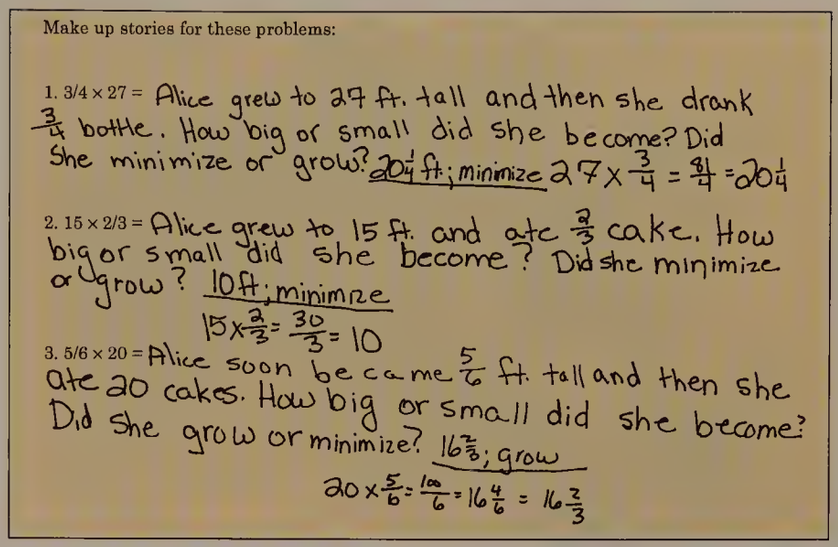
Carrie's question and Nina's and Alan's responses showed that students were beginning to understand the relationship between multiplying by numbers greater than 1 and less than 1. This principle of multiplication is articulated by the Caterpillar in the story. Before meeting the Caterpillar, Alice is frustrated by her inability to control her changes in size. She has no reliable way of predicting whether she will grow larger or smaller. Drinking from the first bottle makes her shrink to 10 inches, but drinking from the bottle she finds in the White Rabbit's house makes her grow too large for the house. Eating one cake enlarges her from 10 inches to over 9 feet tall, but eating the pebble-cakes causes her to shrink small enough so that she can walk through the rabbit's door. Alice begins to be able to control her changes in size after her conversation with the Caterpillar:

> "One side will make you grow taller, and the other side will make you grow shorter."
> "One side of *what*? The other side of *what*?" thought Alice to herself.
> "Of the mushroom," said the Caterpillar. (Carroll 1992, p. 68)

Although a round mushroom cap does not have "sides," Alice copes with the situation by breaking off diametrically opposite pieces of the mushroom. (See the illustration in **fig. 2**.) After experimenting to find out which part of the mushroom makes her grow and which makes her shrink, Alice is able to change her size to suit her purposes: to participate in the tea party and then to unlock the door and enter the garden. Alice's ability to control whether she will grow larger or smaller corresponds to students' understanding that multiplication by a number larger than 1 increases the quantity being multiplied, whereas multiplying by a number less than 1 decreases it.

The worksheet titled "How Tall Is Alice?" gave students a chance to practice multiplication of fractions and to explain their emerging understanding of the relationships between the magnitude of the multiplier and its effect on the starting quantity. The stories that Carmella wrote for the three problems at the end of the worksheet (**fig. 3**) show that she is developing an understanding that multiplication can shrink ("minimize") as well as enlarge ("grow") quantities. Although it is conventional to think of the first quantity in the multiplication expression as the multiplier, it is clear from Carmella's word problems that she is not following that convention. In the first, she describes finding 3/4 of 27, and in the second problem, she finds 2/3 of 15. Her third problem describes finding 20 times as much as 5/6. Nevertheless, her story problems indicate an understanding that multiplication can result in "shrinking" or "growing."

To compare changes in these students' thinking with those of the students in the first class about multiplication problems involving fractions, I gave the students a pretest and a posttest of twenty word problems, ten with fraction multipliers and ten with whole-number multipliers. I compared the solution strategies they used with those employed by the first


Make up stories for these problems:

1. 3/4 × 27 = Alice grew to 27 ft. tall and then she drank $\frac{3}{4}$ bottle. How big or small did she become? Did she minimize or grow? $20\frac{1}{4}$ ft; minimize $27 \times \frac{3}{4} = \frac{81}{4} = 20\frac{1}{4}$

2. 15 × 2/3 = Alice grew to 15 ft. and ate $\frac{2}{3}$ cake. How big or small did she become? Did she minimize or grow? 10 ft; minimize $15 \times \frac{2}{3} = \frac{30}{3} = 10$

3. 5/6 × 20 = Alice soon became $\frac{5}{6}$ ft. tall and then she ate 20 cakes. How big or small did she become? Did she grow or minimize? $16\frac{2}{3}$; grow $20 \times \frac{5}{6} = \frac{100}{6} = 16\frac{4}{6} = 16\frac{2}{3}$

Fig. 3 Carmella's word problems

class of fifth graders on the same pre- and posttests. **Table 1** shows that the students who had discussed *Alice's Adventures in Wonderland* and had learned to represent changes in her size with multiplication used multiplication much more frequently on all the posttest problems, whether the multipliers were whole numbers or fractions. Of the "Alice" students, 75 percent used multiplication to solve all the problems and only 19 percent used division to solve problems with fraction multipliers compared with 28 percent of the "non-Alice" students who used multiplication to solve all problems and 64 percent who divided to solve the posttest problems that had fraction multipliers. Many more "Alice" students had successfully extended their ideas about multiplication to include multiplication by fractional multipliers.

Interviews with five "Alice" students confirmed these changes in their understanding of multiplication. Before the instruction, they said that the only problems that could be solved by multiplication or repeated addition were those with whole-number

TABLE 1

Percent of Students Employing Each Strategy Pattern to Solve Word Problems with Whole-Number and Fraction Multipliers

	Non-Alice Students (n = 25)		Alice Students (n = 21)	
Strategy Pattern	Pretest	Posttest	Pretest	Posttest
Multiplied to solve all problems	4	28	0	77
Multiplied to solve problems with whole-number multipliers and divided to solve problems with fraction multipliers	32	60	62	19
Divided to solve all problems	12	4	5	0
Used a diagram	8	0	10	0
No strategy pattern	44	8	24	5

ILLUSTRATION BY JOHN TENNIEL. FROM *ALICE'S ADVENTURES IN WONDERLAND* BY LEWIS CARROLL

Fig. 4 The Cheshire Cat

multipliers. Students who were able to solve the problems with fraction multipliers used division strategies. See the following problem:

> A carton will hold 42 notebooks. If the carton is 5/6 full, how many notebooks are in the carton?

After instruction, Tamika, like the other interviewed students, said that she would solve the problem by multiplying 42 by 5/6. "Because you're going to find it when it's in sixths and then you'll find the 5. When you times it and you're going to put it into sixths, it's just like dividing it into sixths and then you get the 5 sixths," she said. She wrote,

$$42 \times \frac{5}{6} = \frac{210}{6} = 35.$$

Her explanation indicates that she understood that 5/6 of a quantity could be found by multiplying by 5/6 and that she understood the composite nature of the multiplier 5/6; it simultaneously partitions the quantity into sixths and collects 5 of them.

Alice's changes in size thus provide students with the opportunity to develop an understanding of how multiplication by fractions can symbolize multiplicative situations in which the product is smaller than the original quantity. It can also extend their knowledge of whole-number multiplication to include multiplication by numbers less than 1. Other incidents in the story can afford opportunities to develop additional elegant, yet fundamental, concepts related to multiplication and proportionality.

Multiplication by a very small number. The idea of a limit or how small a product can be can also be discussed. In chapter 2, Alice discovers that fanning herself is causing her to shrink rapidly, and she drops the fan "just in time to save herself from shrinking away altogether" (Carroll 1992, p. 22). Alice fears that if she gets to be too small, she will cease to exist. Ask students to find Alice's height if it were multiplied by a very small fraction. Students might suggest multiplying by 1 one-millionth or 1 one-billionth. This discussion gives the class a chance to explore informally ideas such as whether there is a smallest fraction and whether Alice would ever go out of existence and completely vanish. Although Alice might become too small to see with the unaided eye, she would not vanish unless her height were multiplied by 0.

The distinction between additive/subtractive change and multiplicative change. Asking students to compare and contrast the way in which the Cheshire Cat appears and disappears with the changes in Alice's size provides an opportunity to discuss the distinctions between additive and multiplicative change. Whether Alice shrinks or grows taller, she is always a complete girl with head, hands, feet, and so on. The Cheshire Cat, however, appears and disappears part by part (see **fig. 4**). In chapter 6, the cat disappears bit by bit until only the grin remains. In chapter 8, the grin appears first, but Alice decides not to speak to the cat until its ears are seen. The Cheshire Cat can also disappear completely. His transformations from visibility to invisibility result from the additive operation of subtraction, in contrast to Alice's transformations, which are multiplicative. It is interesting to note that some characters in the story, such as the baby who becomes a pig and the playing cards, are transformed in ways that cannot be represented mathematically.

Multiplicative relationships at the heart of proportionality and similarity. Multiplicative change also underlies the concepts of proportionality and similarity in the context of geometric figures. Sometimes Alice's changes in size preserve her proportions, but other changes are disproportionate, as when she says she is "opening out like the largest telescope that ever was" (Carroll 1992, p. 15); the illustration in chapter 2 shows her as disproportionately taller (**fig. 5**). Students could further explore proportionality, similarity, and the distinction between additive and multiplicative change by comparing pairs of rectangles like those in **figure 6**, stating whether the relationship between the rectangles in each pair is additive or multiplicative and whether the rectangles are similar. The two rectangles in **figure 6a** are similar because for both pairs, the sides of one rectangle are 5/3 (or 3/5) the length of the sides of the other rectangle. The rectangles in **figure 6b** are not similar. The sides of the smaller rectangle are 2 units less than the sides of the larger rectangle.

ILLUSTRATION BY JOHN TENNIEL, FROM *ALICE'S ADVENTURES IN WONDERLAND* BY LEWIS CARROLL

Fig. 5 Alice grows out of proportion.

The rectangles in **figure 6c** are not similar because the length has been increased by a factor of 2, whereas the width has been increased by a factor of 3, or the sides of the larger rectangle are 6 units longer than the sides of the smaller rectangle. Extending the discussion of additive and multiplicative relationships to geometric figures will help students further develop the concepts of proportionality that are common to numeric and geometric contexts.

Alice's Adventures in Wonderland can provide a rich environment for mathematics and language learning. Reading the story while learning about topics such as multiplication of fractions, multiplicative change, proportional reasoning, or similarity provides a context of imaginative representation in which students can explore, discuss, and deepen their understanding of mathematics. Focusing on the mathematical aspects of the story will also help students analyze and gain a deeper appreciation for the literary qualities of the story.

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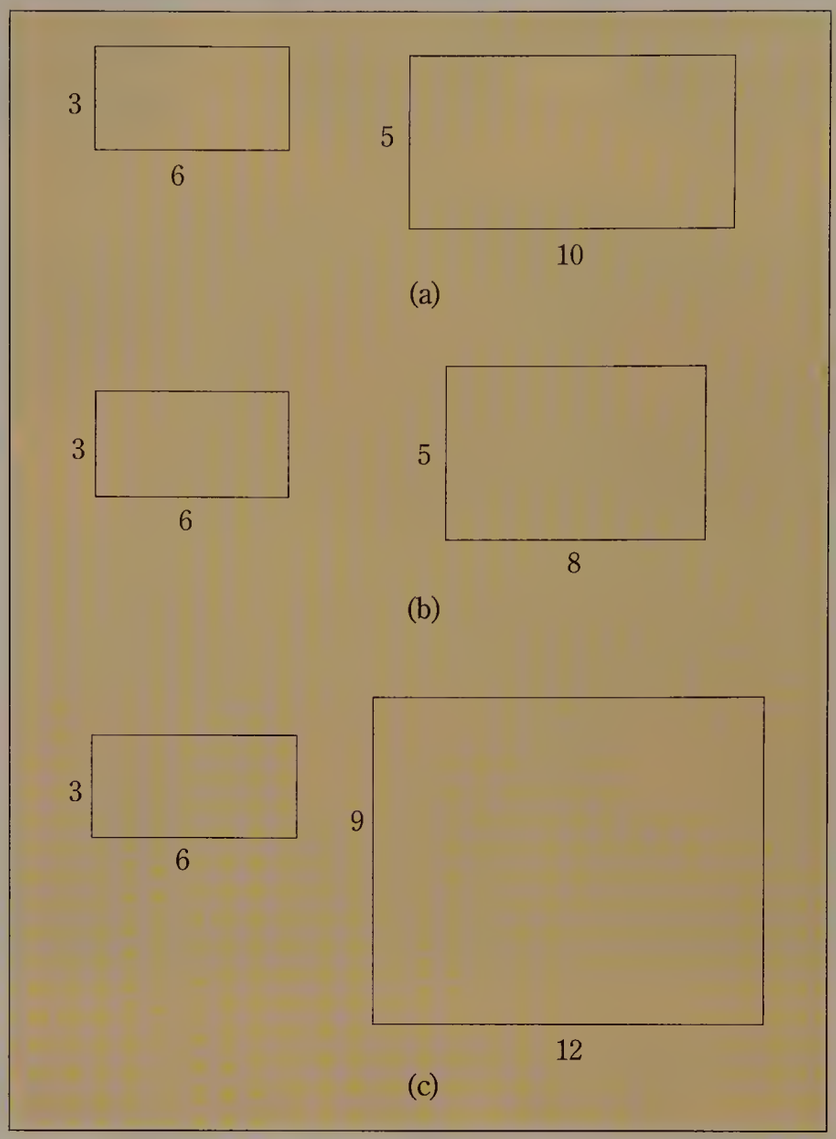

Fig. 6 Similar and nonsimilar rectangles

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For more articles relating mathematics and literature, see the April 2005 focus issue of MTMS, *titled "Connecting Mathematics and Literature in the Middle Grades."—Ed.*

(See the worksheet on the next page.) □

How Tall Is Alice?

NAME _____

Show what happens to Alice's height when she eats cake or drinks potions.

A Cake That Makes Her **6 Times** as Tall		A Potion That Makes Her 1/6 as Tall	
Starting Height	Equation	Starting Height	Equation
4 feet		4 feet	
8 feet		8 feet	
2 feet		2 feet	
3/4 foot		3/4 foot	
3/8 foot		3/8 foot	

A Potion That Makes Her **5/3 Times** as Tall		A Cake That Makes Her 3/5 as Tall	
Starting Height	Equation	Starting Height	Equation
10 feet		10 feet	
12 feet		12 feet	
8 feet		8 feet	
2 feet		2 feet	
6 feet		6 feet	

If Alice drinks a potion that multiplies her height by 7/8, will she become taller or shorter? Why?

Make up stories for these problems:

1. $3/4 \times 27 =$

2. $15 \times 2/3 =$

3. $5/6 \times 20 =$

From the December 2006/January 2007 issue of **Mathematics Teaching in the Middle School**

Engaging Contexts for the Game of Nim

CHARLES A. (ANDY) REEVES AND ROSEMARIE REEVES GLEICHOWSKI

MIDDLE SCHOOL TEACHERS REALIZE the value of students playing games in mathematics classes if those games emphasize problem-solving strategies, algebraic reasoning, or spatial sense. At the end of an action-research project that investigated the value of game playing, Lach and Sakshaug (2005) reported:

> I saw evidence that students learned how to strategize better and how to explain their reasoning about strategies. I saw an increase in the number and length of discussions over the course of the study from the beginning to the end. Students also showed more confidence in their own ability to reason and to communicate about how to solve problems. (p. 175)

The study discussed above used nine games that emphasized different problem-solving strategies. We will report here on our use of one game, nim, and its value in teaching students the strategy of *working backward*. This particular strategy becomes increasingly valuable as students approach their first algebra course. Working backward is the primary method for solving equations by undoing algebraic

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expressions using inverse operations.

The game of nim dates back to ancient times and appears in various formats throughout history. Nim is a two-person game that uses a certain number of objects. Each player in turn can take 1, 2, or 3 objects, and then it is the other player's turn. The person who takes the last object or forces the opponent to take the last object, depending on the version being played, is the winner. Some versions of the game arrange the objects in rows, with an additional rule that you can only take objects from one row at a time. Nim has always been popular because the rules are easy to understand and there are no complicated game boards or pieces to purchase.

Nim has also entered the electronic age, and various versions can be found on the Web. These e-games vary in terms of the number of objects that begin the game and the arrangement of the objects in rows. A Web search for nim puzzles will produce a number of options for nim enthusiasts.

Survivor Thailand: An Engaging Format

IN 2002, THE *SURVIVOR: THAILAND* TELEVISION show aired a contest between its two competing tribes, Sook Jai and Chuay Ghan. The episode's rules were similar to nim. Twenty-one flags were stuck in the ground, and a team could remove either 1, 2, or 3 flags (see **fig. 1**). The team that took the last flag won. We recommend showing that segment of the show to a class of students as initial motivation, and then having the students work in pairs and play the game using toothpicks. (See a description at the end of the article on how we obtained a copy of the tape.) The objective is for students to discover and describe a winning strategy.

REPRODUCED FROM THE *SURVIVOR: THAILAND* SHOW BY PERMISSION OF CBS; ALL RIGHTS RESERVED

Fig. 1 Twenty-one flags are stuck in the sand to begin the *Survivor: Thailand* game.

Middle school students usually discover that if they can leave their opponent with 4 flags, they can win the game. (If your opponent takes 3 flags, then you take the last flag; if your opponent takes 2, then you take 2, including the last flag; if your opponent takes 1, then you take 3, including the last flag.) When students describe this winning tactic, they will have worked backward one time. (See the student work in **fig. 2**.) However, to continue working backward using this same reasoning is difficult for students. They have already used the strategy of leaving an opponent with 4 flags. If that person leaves an opponent with 8 flags, he or she can also win; this is the second step in working backward with a winning strategy. How will they leave an opponent with 8 flags? Back it up a third time. If they leave 12 flags, then on their next turn, they can leave 8. Back it up one more time. If they leave 16 flags, then they can leave 12. Then the clincher step occurs. If they leave 20 flags, then they can leave 16. So those who go first and take 1 flag can win the game. Finding the winning strategy, therefore, requires using the process of working backward five times. In working with two classes of eighth-grade Algebra 1 Honors students, almost all could see the last-step strategy but only about half could back it up to the beginning. (In **fig. 3**, students play the game with toothpicks and describe a successful strategy from the start.)

Simplifying the Game

TEACHERS MIGHT WANT TO START WITH A SIMpler but similar game, depending on the class of students. Simpler versions of nim involve starting with fewer objects initially or removing fewer objects with each turn. "Poison apple" is one such well-known game, but we called it "rotten apple" with students because of the negative connotation of the word *poison*. To play "rotten apple," students start with 14 red chips and 1 yellow chip, which represent 14 good apples and 1 rotten apple. When it is their turn, students can take 1, 2, or 3 apples. The object is to force an opponent to get the yellow (rotten) apple. Students find that they can win by leaving an opponent with 5 apples, including the rotten apple, because on the next turn, they can then leave their opponent with the rotten apple. Backing it up one more step, they can leave their opponent with 9 apples to ensure that they leave them with 5. Again, backing up the reasoning process, they can leave them with 13 apples to ensure that the opponent is left with 9. So the player who goes first and takes 2 apples can win the game.

The game can be simplified even further by using a carton of eggs. Two games, "golden egg" and

"rotten egg," start with a dozen eggs and, depending on the game, one egg is labeled "golden" or "rotten." The goal of "golden egg" is to get the last egg (a player cannot take the golden egg until last). The goal of "rotten egg" is to force the opponent to take the last (rotten) egg. In the latter version, a player can force an opponent to get the last egg by leaving 5 eggs, including the bad one. Backing it up again, leaving 9 eggs will ensure that 5 are left with the opponent, so a player can win if he or she goes first and takes 3 eggs. For "golden egg," if an opponent is left with 4 eggs, he or she cannot get the golden egg on one turn so will not win. A player can ensure leaving an opponent with 4 eggs by leaving 8 eggs on the previous turn.

How does one leave 8 eggs? Let the opponent go first; if he or she takes 1, the other player takes 3; if 2 are taken, the other player takes 2; and if 3 are chosen, the other player takes 1. Notice that of the games described, this is the first that gives a player the advantage if an opponent goes first.

An even simpler version of nim is described by Sakshaug, Olson, and Olson (2002) and is called "take two." The game is played by starting with 7 chips. Players take 1 or 2 chips each turn, and the winner is the one who gets the last chip. The authors report this result for one student:

> In a recent experience of playing this game with a kindergartner named Nicholas, we used pennies. The first time we played the game, when we got down to three pennies left and it was my turn, Nicholas began laughing. I asked what he was laughing at, and he said, "I'm going to win!" I asked how he knew that, and he said, "If you take one, I get two. If you take two, I get one." After playing three times, Nicholas began counting how many pennies were left before he took his turn and would determine whether he could take enough to leave three. After playing five times, he suggested that we play "take three or less." He immediately took three pennies and grinned at me. When I asked why he was smiling, he explained that he had won. (Sakshaug, Olson, and Olson 2002, pp. 56–57)

The authors continued playing with Nicholas and found that he had a partial grasp of a winning strategy, and he was on his way to a full understanding. He was intuitively *working backward* in that he sensed what result would give him the win and what conditions prior to that end were advantageous.

What Is the Value of Playing Nim?

ALL THESE VERSIONS OF NIM ARE ENJOYABLE to students, but in these days of accountability, we should be able to document that students are learning something that is mathematically valuable. Berkman (2004) states:

> A simple game like "poison apple," in which each child takes turns removing one or two apples from a tree until one person is stuck with the last apple, helps develop planning, strategizing, and pattern-recognition skills. (p. 247)

However, we wanted to learn if the algebra students with whom we used the *Survivor: Thailand* and "rotten apple" games could see a connection between those games and solving equations. As stated in *Principles and Standards for School Mathematics* (NCTM 2000):

> School mathematics experiences at all levels should include opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics. These connections can be to other subject areas and disciplines as well as to students' daily lives. (pp. 65–66)

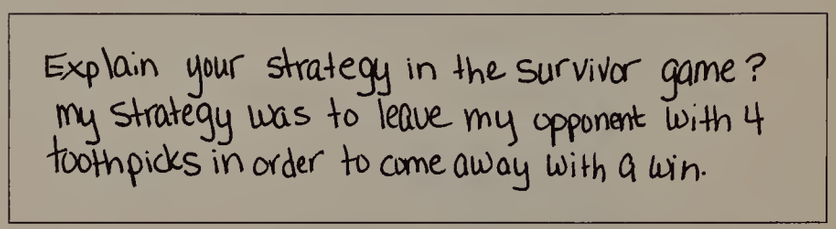
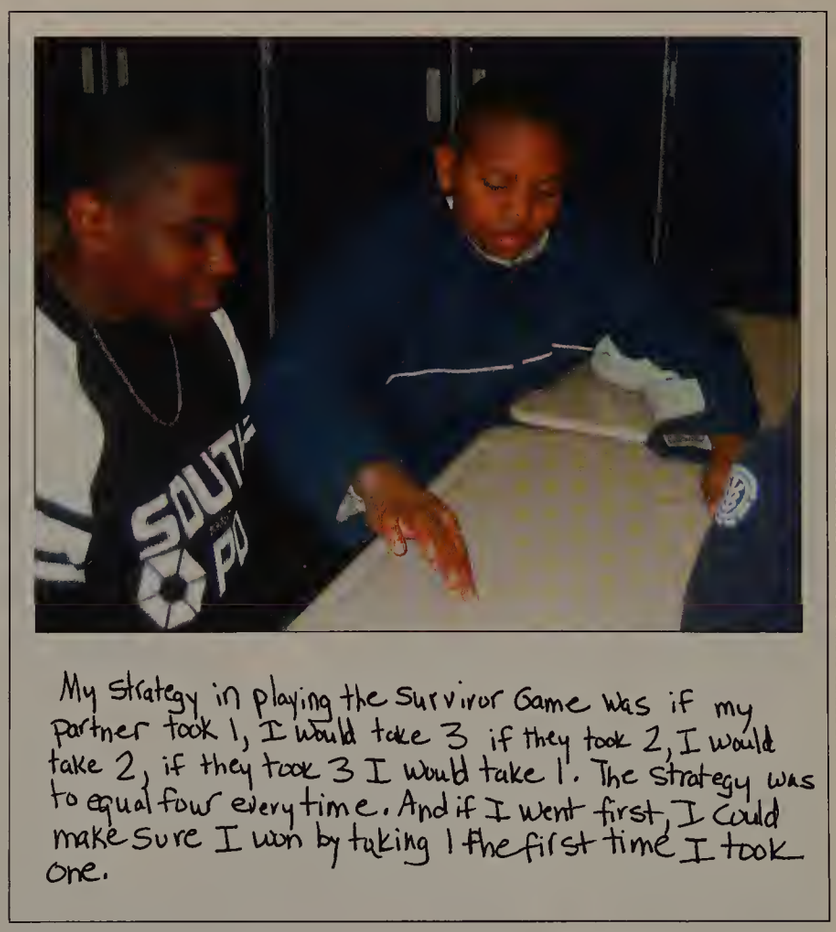

Explain your strategy in the survivor game?
my strategy was to leave my opponent with 4 toothpicks in order to come away with a win.

Fig. 2 This excerpt shows that the student worked backward one time to win the *Survivor: Thailand* game.


My strategy in playing the Survivor Game was if my partner took 1, I would take 3 if they took 2, I would take 2, if they took 3 I would take 1. The strategy was to equal four every time. And if I went first, I could make sure I won by taking 1 the first time I took one.

PHOTOGRAPH BY ROSEMARIE REEVES GLEICHOWSKI; ALL RIGHTS RESERVED

Fig. 3 Students play the *Survivor: Thailand* game with toothpicks, and student work shows a "total mastery" strategy.

We asked students, "How are these games similar to solving equations?" One typical response is included as **figure 4**. Almost all the students realized that *working backward* was the key to both playing these games successfully *and* solving equations by undoing algebraic expressions one step at a time.

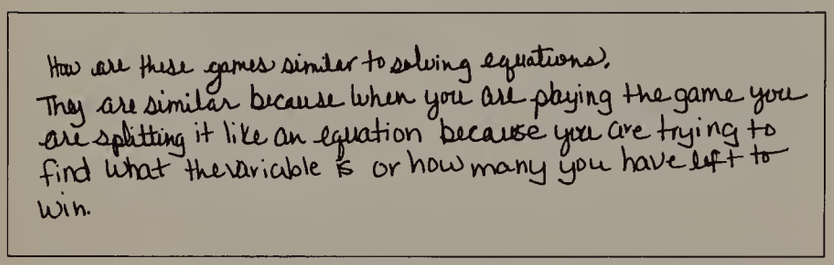

How are these games similar to solving equations?
They are similar because when you are playing the game you are splitting it like an equation because you are trying to find what the variable is or how many you have left to win.

Fig. 4 A student explains the connection between nim and solving equations.

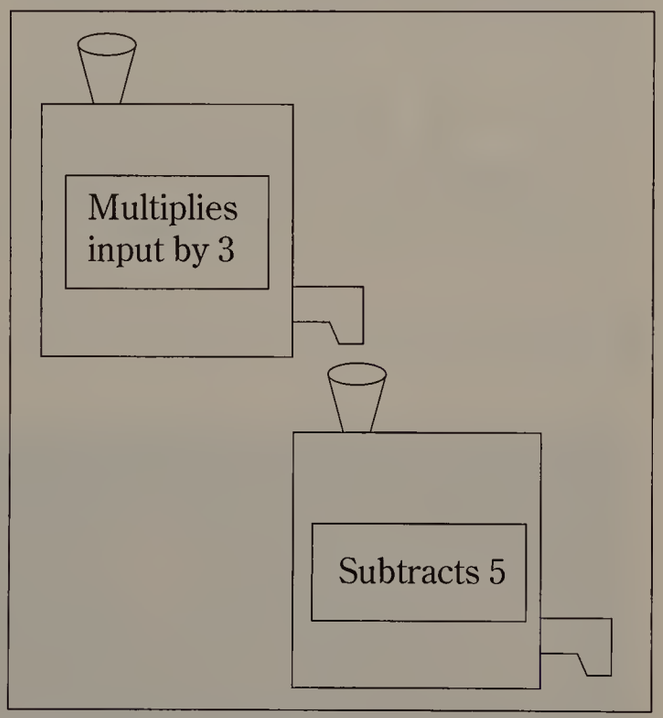


Fig. 5 Two function machines are set up to show $3x - 5$.

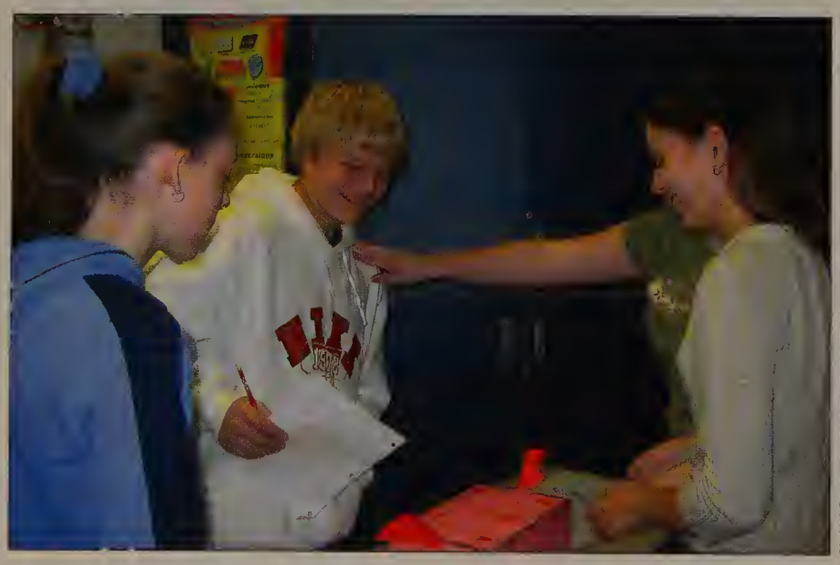


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Fig. 6 Students try their homemade function machines on each other.

Using Nim to Complement Other Exposures to *Working Backward*

THESE STUDENTS HAD PREVIOUSLY EXPERIenced other situations that required working backward in their regular mathematics curriculum. Textbook companies have long included word problems for which working backward is appropriate. An example is provided below:

> The new movie that opened yesterday was so boring that half of the people in the theater left after the first 45 minutes. In the next 15 minutes, half of the remaining people left. After 18 more people left, there were 35 people left who were there at the start of the movie. How many people were there at the start? (Pearson Prentice Hall 2004, p. 219)

Students are encouraged to make a flowchart of the action so that they can visualize reversing the order of the operations and use inverse operations to work backward from the end result to find the beginning condition. The starting condition for this problem is 212 people.

Another exposure to the process of working backward comes from function machines. Students learn to undo an expression, such as $3x - 5$, by taking output numbers and running the machines backward to find the input. Again, visualizing what happens in the machine, as shown in **figure 5**, helps students understand how the process works. **Figure 6** shows some middle-grades algebra students using their homemade version of a function machine to practice "guess my rule" and then finding input numbers by working backward.

Summary

THE VARIOUS VERSIONS OF NIM DESCRIBED IN this article are enjoyed by students and also help them learn how to work backward. When combined with other activities to develop this strategy, students find that they are well on their way to learning how to solve equations by undoing expressions one step at a time. The connections that students see between mathematics and real-world applications are probably just as valuable as the skill itself. The time spent is quite enjoyable and has a definite payoff down the road of mathematics *and* down the road of life.

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How we obtained a copy of the *Survivor: Thailand* videotape: We called our local CBS affiliate and asked for the person who handles permissions. We told her that we taught mathematics and wanted to use the tape with our classes. We agreed verbally not to use the tape in any other way and offered to send her a blank tape. Under those conditions, she sent us a copy of the tape. □

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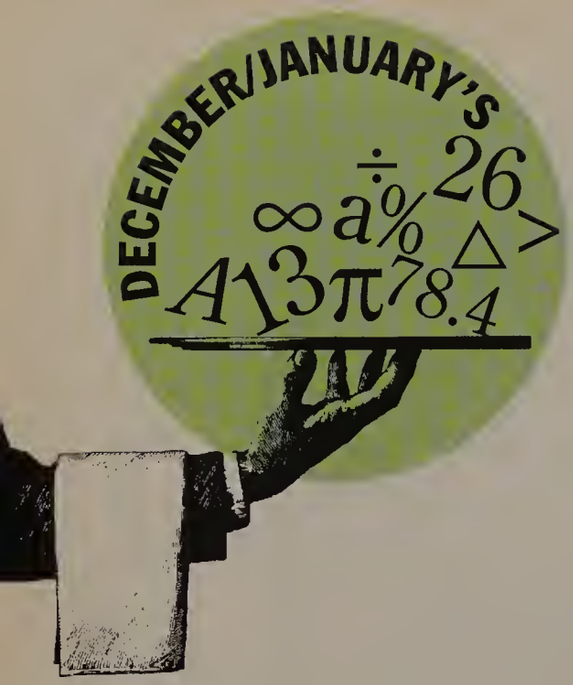
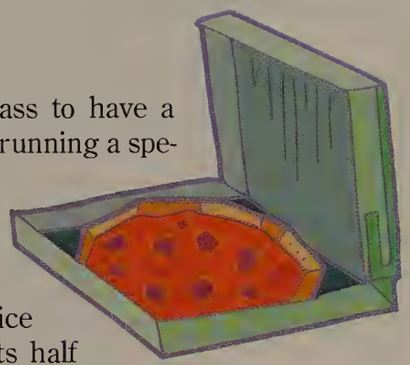


Menu of

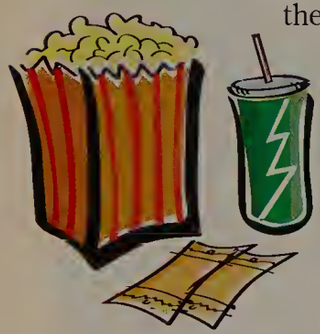


1. Your teacher allows the class to have a pizza party. The pizza parlor is running a special: The first large pizza in an order costs \$15.99, the second pizza (in the same order) costs half the price of the first, the third pizza costs half the price of the second, the fourth costs half the price of the third, as so on. If your class can only buy whole pizzas and has exactly \$31.75 to spend (excluding tax and tip), what is the greatest number of pizzas that the class can purchase? Explain your reasoning.

2. Use only addition and multiplication symbols among the digits on the left side of the equation to make this mathematical sentence true.

$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9 = 100$$

3. While at the movie theater, you decide to buy some hot buttered popcorn and a cold soda. You notice that the popcorn costs \$2 plus half its price and the soda costs \$1 plus half its price. If you buy one bag of popcorn and one soda, what will be the total cost, excluding sales tax? Be careful; this problem is tricky.



4. A *palindrome* is a number that reads the same both forward and backward, such as 22, 343, and 1441. Exactly how many whole-number palindromes exist between 100 and 200?

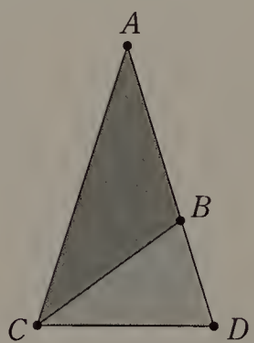
5. Find the difference between the least and greatest prime factors of 33,660.

6. For the following addition problem, A, B, C, and D represent different whole numbers. Determine only the value for the letters C and D.

$$\begin{array}{r} A\,A\,A \\ +\,B\,B\,B \\ \hline D\,D\,D\,C \end{array}$$

7. You meet three men at a meeting: Mr. Yellow, Mr. Green, and Mr. Blue. They are wearing yellow, green, and blue bow ties. Mr. Yellow says, "Did you notice that the color of our ties are different from our names?" The person in the green bow tie says, "Yes, you are right!" What color of tie is each man wearing?

8. Observe $\triangle ABC$. $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$. If $\overline{BC}$ bisects $\angle ACD$, find the degree measure of $\angle CAD$.



Problems

9. If the pattern below continues, what will be the 1002nd letter?

MATHISFUNMATHISFUNMATHISFUN....

10. A census was taken in 2000, comparing populations from a 1990 census. The population of Newtonville increased by 20 percent, whereas the population of Kepler City decreased by 10 percent. It was also discovered that in 2000 the two populations were equal. What percent of the 1990 population of Kepler City was the 1990 population of Newtonville?

11. Imagine that a new operational symbol for mathematics has been developed. This symbol is ~ and is represented by

$$a \sim b = \frac{ab}{a-b}.$$

Using this new symbol, find the value of $(2 \sim 3) \sim 4$.

12. Using a constant rate, you can saw a log into 3 pieces in 12 minutes. At the same constant rate, how long will it take you to cut a similar log into 6 pieces? Write an explanation of how you found your answer.

13. Doug is working on a mathematics problem in class. By mistake, he multiplies by 10 then divides by 5 instead of multiplying by 5 and then dividing by 10. If the incorrect answer is 3, what is the correct answer?

14. How many numbers between 9 and 100 exist such that the product of the digits is greater than the actual number?

15. A high-rise apartment building has four elevators. Each elevator travels upward at an average rate of 18 feet per second. The elevator travels 12 feet from one floor to the next. Suppose you get on the elevator on the first floor and travel straight to the fiftieth floor without stopping. To the nearest second, how long will it take to travel from the first floor to the fiftieth floor on this elevator?

16. Little green frogs can catch 3 flies in 1 hour; big bullfrogs can grab 8 flies each hour. In an average day, bullfrogs sleep 17.5 hours, and little green frogs only need to sleep 7 hours. After a full 3 days, which frog can catch more flies, the bullfrog or the little green frog? How many more flies did that frog catch after 3 days?

(*Solutions on pages 258–59*)

Prepared by DAVID ROCK, *rock_david@colstate.edu, Columbus State University, 4225 University Avenue, Columbus, GA 31907*

MTMS readers are encouraged to submit single problems or groups of problems by individuals, student groups, or mathematics clubs to be considered for publication. Send to the "Menu" editor, DAVID ROCK, at rock_david@colstate.edu. *MTMS* is also interested in students' creative solutions to these problems. Send these items to "The Thinking of Students" editor, EDWARD MOONEY, at mooney@ilstu.edu. Both problems and solutions will be credited. For additional menu problems, see the NCTM publication, *Menu Collection: Problems Adapted from "Mathematics Teaching in the Middle School"* (stock no. 726).

Mathematics Teaching in the Middle School

Solutions to "December/January's Menu of Problems"

(*Continued from pages 256–57*)

(*Alternative approaches to those suggested here are encouraged.*)

1. 7 pizzas. Pizza 1 costs \$15.99; pizza 2 costs \$7.995, which is rounded to \$8.00 in U.S. currency; pizza 3 costs \$4.00; pizza 4 costs \$2.00; pizza 5 costs \$1.00; pizza 6 costs \$0.50; pizza 7 costs \$0.25 (rounding each time) for a total of \$31.74.

2. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$. There may be other solutions. With $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$, the order of operations requires multiplication to be completed before addition, so 8×9 must be done first, yielding $1 + 2 + 3 + 4 + 5 + 6 + 7 + 72$. Adding the remaining numbers gives 100.

3. \$6.00. The popcorn (p) costs \$2.00 plus half its price, so \$2.00 must be the "other" half of the \$4.00 price. In a similar manner, \$1.00 is half the price of the soda, so the soda costs \$2.00. You can also solve it algebraically: $p = \$2 + (1/2)p$.

4. There are ten palindromes. They are 101, 111, 121, 131, 141, 151, 161, 171, 181, and 191. Since the number is between 100 and 200, the hundreds place must be a 1. Since we are looking for palindromes, and 1 is in the hundreds place, then the ones place must also be a 1.

5. 15. The prime factorization of 33,660 is $2 \times 2 \times 3 \times 3 \times 5 \times 11 \times 17$. The difference between 17 and 2 is 15.

6. $C = 0$, and $D = 1$. A and B can have several possible solutions, but C and D can only be 0 and 1, respectively. If $A + B < 10$, then D would have to equal C, which cannot happen. When $A + B > 10$, then $D = C + 1$ cannot be possible, since the leftmost D will have to be 1. If $A + B = 10$, then $D = 1$ and $C = 0$.

7. Of the three men, Mr. Yellow is wearing the blue tie, Mr. Green is wearing yellow, and Mr. Blue is wearing green. The reason Mr. Yellow is wearing the blue tie and not the green tie is because in the passage, Mr. Yellow is talking to the man wearing the green tie. Therefore, Mr. Yellow must be wearing the blue tie. The person wearing the green tie cannot be wearing the blue tie so he must be wearing the yellow tie. Therefore, Mr. Blue is wearing the green tie.

8. 36°. Since $\overline{BC}$ bisects $\angle ACD$, $\angle ACB \cong \angle BCD$. Let the measures of these angles equal x. Since $\triangle ABC$ is isosceles, the measure of $\angle BAC = x$. Since $\triangle BCD$ is isosceles as well, $\angle CBD \cong \angle BDC$. Let the measures of these angles be y. Since the sum of the angles of a triangle is 180°, the measure of $\angle ABC = 180 - 2x$. But the measure of $\angle ABC = 180 - y$, so $2x = y$. In $\triangle BDC$, $2y + x = 180$. Substituting yields $2(2x) + x = 180$, or $5x = 180$, or $x = 36$.

9. T. The pattern contains 9 letters; $1002 \div 9 = 111$, remainder 3. Therefore, you could spell MATHISFUN 111 complete times with 3 letters left over. The 1002nd box would be T.

10. 75 percent. If N and K represent the 1990 populations of Newtonville and Kepler City, respectively, $1.2N = 0.9K$, so:

$$\frac{N}{K} = \frac{0.9}{1.2}$$
$$\frac{N}{K} = \frac{9}{12}$$
$$\frac{N}{K} = \frac{3}{4}$$
$$\frac{N}{K} = 0.75$$
$$\frac{N}{K} = 75\%.$$

11. 2.4. $2 \sim 3 = (2 \times 3)/(2 - 3) = -6$, and $-6 \sim 4 = (-6 \times 4)/(-6 - 4) = -24/-10 = 2.4$.

12. 30 minutes. Cutting a log into 3 pieces requires 2 cuts. If the time allowed is 12 minutes and the rate is constant, each cut takes 6 minutes. Cutting a log into 6 pieces requires 5 cuts. At 6 minutes per cut, the total time would be 30 minutes.

13. 0.75. First you need to determine the number that Doug begins with. If A is the number, then $(A \times 10)/5 = 3$. Solving for A yields $A = 1.5$. You are looking for $(A \times 5)/10$, so $(1.5 \times 5)/10 = 0.75$.

14. None. In an attempt to get larger products, begin by using every number ending in 9: 19, 29, 39, 49, 59, 69, 79, 89, and 99. You will see that it is impossible. For example, $A9$ would be $A \times 9$, which will always be less than $A0$, since $A0$ is $A \times 10$. Therefore, if $A9$ does not work, neither will $A8$, $A7$, $A6$, etc.

15. 33 seconds. First, you must determine how many floors you are traveling. Going from the first to the fiftieth floor is not 50 floors but only 49. (Think of it this way: How many floors do you go up when going from the first floor to the second floor? Only one floor.) You travel 12 feet for each floor: $49 \times 12 = 588$ feet. If the elevator travels an average of 18 feet per second, then $588/18 = 32.67$, or 33 seconds, since the problem says "to the nearest second."

16. The bullfrog catches a total of 3 more flies after 3 days than the little green frog. The bullfrog is awake for 6.5 hours each day and eats 8 flies per hour for a total of

52 flies per day and 156 flies after 3 days. The little green frog is awake 17 hours and eats 3 flies per hour for a total of 51 flies per day. Therefore, the bullfrog catches 1 more fly each day for a total of 3 more flies. □

Solutions to December 2006/January 2007 "Cartoon Corner"

(*Continued from page 262*)

100th Birthday Party

1. Herman is 50 years old. Three big candles at 15 years each is 45 years, and 5 small candles at 1 year each is 5 years. Thus, 45 + 5 = 50.

2. Each small candle represents 2 years. If Herman is 100 years old and each of the 3 big candles stands for 30 years instead of 15, then the 5 small candles would have to make up the difference of 100 – 90 = 10 years; 10 ÷ 5 = 2.

3. The equation is $3b + 5s = 100$. Substituting 30 for b and 2 for s gives $3 \times 30 + 5 \times 2 = 90 + 10 = 100$.

4. Making a chart or list is one way to solve the equation. Encourage students to search in an organized way, such as starting with the largest value for b and working down to the smallest value. Assume that $b > s$, since the bigger candle would usually represent more years than a smaller one. The four solutions are (30, 2); (25, 5); (20, 8); (15, 11). Students might also find these solutions by graphing or by guess, check, and revise.

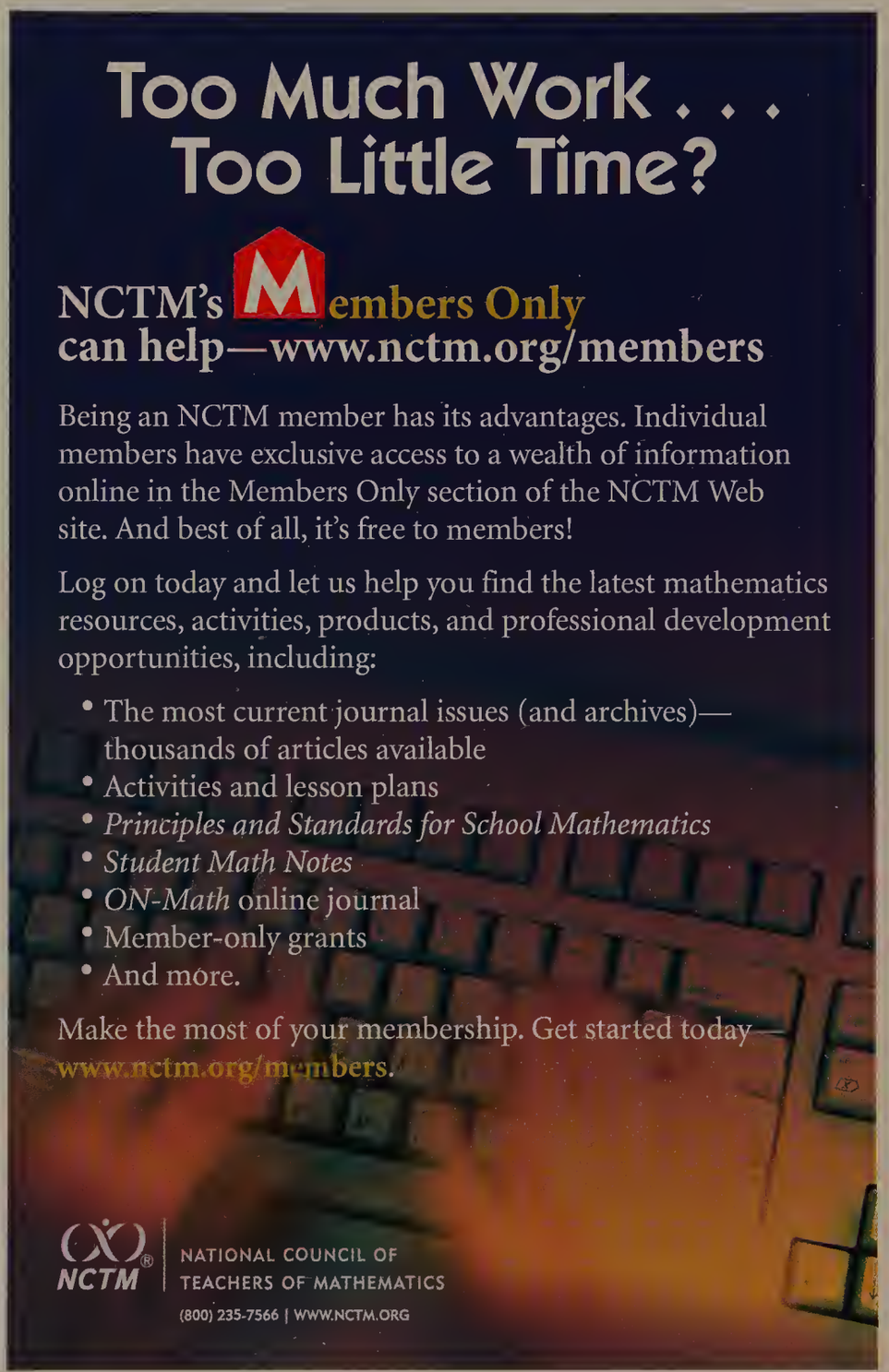
Time Flies!

1. At age 13, a year would be 1/13, or approximately 7.7 percent of a student's life. At 12, a year would be 1/12, or approximately 8.3 percent. So a year would seem 8.3% – 7.7%, or 0.6%, shorter.

2. Students will compare 1/20 and 1/10 as fractions and percents. As fractions, using 20 as the common denominator, 1/10 – 1/20 = 2/20 – 1/20 = 1/20. As a percent, 1/10 = 10% and 1/20 = 5%, and 10% – 5% = 5%. In real-time terms, a year would seem like 1/20 of a year shorter, and 1/20 × 365 days is 18 1/4 days, or 18 days, 6 hours. A day would seem like 1/20 × 24 hours shorter, or 1 hour, 12 minutes.

3. At age 2, using the cartoon's logic, a year would be 1/2 of your life so far.

4. Answers will vary. It is hoped that students will return to school with various stories of how believable (and funny, or not so funny) the situation is. The important part of this question is that, in explaining it to others, the students would learn more about comparing fractions and percents. □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CANDIDE WALTON AND TAMELA HANEBRINK

Talk Isn't Cheap

THE RICHARDSON FAMILY IS SHOPPING. Lacy has found the perfect dress for a dance, and Kyle has located a video game. However, they have a problem. Neither of them has any money, nor do they know exactly which store their parents are in. The solution? Cell phones for everyone. Currently, only the parents have cell phones. If every member of the Richardson family, a family of four, had cell phones, then the children could contact their parents. Later, the family decides that, in general, cell phones for all will be beneficial for parents and children.

The family begins investigating a new plan. At the present time, the parents have separate rate plans, but they soon discover that it will be cheaper to have one rate plan with three additional lines, instead of adding lines to the two separate plans. Based on that information, they research Express Wireless and Flash Cellular, the only two wireless companies that serve their area. They are in the process of choosing the best plan for their family, based on the number of minutes they have used during the past twelve months under their current plans. They are including with those minutes an estimate of the additional minutes that might be used by the two children. **Table 1** shows their calculations. Based on this information, they are considering the two plans in **table 2**.

The Richardsons note that both companies provide the same coverage areas and offer the same services, such as nationwide long distance (no roaming or long-distance charges), voice mail, call forwarding, call waiting, three-way calling, caller ID, and so on.

TABLE 1
The Richardsons' Wireless Phone Log in Minutes during the Past 12 Months, Containing an Estimate of Possible Usage by Their Children

Month	Daytime (7:00 A.M.–6:59 P.M. Mon.–Fri.)	Nights (7:00 P.M.–8:59 P.M. Mon.–Fri.)	Nights (9:00 P.M.–6:59 A.M. Mon.–Thurs.)	Weekends (9:00 P.M.–6:59 A.M. Fri.-Mon.)
1	652	163	63	211
2	700	108	71	264
3	637	195	42	196
4	720	94	65	147
5	586	137	56	214
6	681	118	37	201
7	725	65	49	183
8	556	244	68	238
9	697	112	57	119
10	740	51	53	175
11	732	115	55	183
12	588	253	66	164

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TABLE 2
The Two Cell-Phone Plans

Plan Options	Express Wireless Family Calling Plan (Plan E)	Flash Cellular Family Calling Plan (Plan F)
Monthly rate	$69.95 per month	$69.99 per month
Additional lines (up to 3)	$20.00 per additional line	$9. 99 per additional line
Shared anytime minutes	700	700
Unlimited nights and weekends	$14.95	—
Nights	Mon.–Thurs. beginning at 7:00 p.m., ending at 6:59 a.m.	Mon.–Thurs. beginning at 9:00 p.m., ending at 7:00 a.m.
Weekends	Fri.–Mon. beginning at 7:00 p.m., ending at 6:59 a.m.	Fri.–Mon. beginning at 9:00 p.m., ending at 6:59 a.m.
Additional minutes	$0.40 per additional minute	$0.45 per additional minute
Note:	Customers *do not* include minutes used from 7:00 p.m.–8:59 p.m. Mon.–Fri. as part of their shared 700 anytime minutes	Customers *must* include minutes used from 7:00 p.m.– 8:59 p.m. Mon.–Fri. as part of their shared 700 anytime minutes

Questions

In questions 1–3, disregard any taxes, surcharges, fees, government service charges, and so on that would normally be included in the bill.

1. Complete the chart below to calculate the Richardsons' bill for each month listed under each plan.
2. Based on the twelve-month phone log information, explain which plan appears to be the most economical for the Richardsons.
3. How does the difference between the two plans regarding the time intervals for nights and weekends affect the bill under each plan?

(*Answers on page 288*)

Month	Plan E	Plan F
Month 1		
Month 2		
Month 3		
Month 4		
Month 5		
Month 6		
Month 7		
Month 8		
Month 9		
Month 10		
Month 11		
Month 12		

Cartoon Corner

Edited by ANDY REEVES, *areeves@stpt.usf.edu, who is a professor at the University of South Florida—St. Petersburg, St. Petersburg, FL 33701, and* MARY LOU BEASLEY, *fsuseminole@prodigy.net, who is a math coach at Lealman Intermediate School, St. Petersburg, FL 33714.*

Herman *by Jim Unger*

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"The big ones are 15 years."

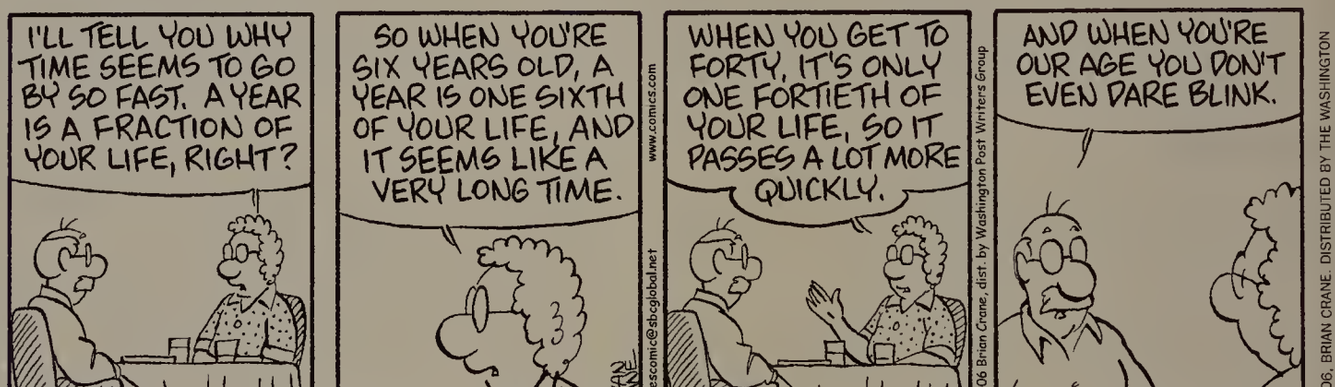
100th Birthday Party

Note: This cartoon helps celebrate the 100th birthday of NCTM's journal for grades 8–14, the *Mathematics Teacher.*

Discussion questions

1. If the small candles each represent 1 year, how old is Herman?
2. If the big candles represent 30 years each, and Herman is 100 years old, how many years does each small candle represent?
3. Write an equation using b for "number of years for each big candle" and s for "number of years for each small candle," which would mathematically illustrate a cake when Herman is 100 years old. Show that your equation works when $b = 30$ and $s = 2$, as described in problem 2.
4. Find all the whole-number values for b and s that solve your equation in question 3.

Pickles *by Brian Crane*


I'LL TELL YOU WHY TIME SEEMS TO GO BY SO FAST. A YEAR IS A FRACTION OF YOUR LIFE, RIGHT?
SO WHEN YOU'RE SIX YEARS OLD, A YEAR IS ONE SIXTH OF YOUR LIFE, AND IT SEEMS LIKE A VERY LONG TIME.
WHEN YOU GET TO FORTY, IT'S ONLY ONE FORTIETH OF YOUR LIFE, SO IT PASSES A LOT MORE QUICKLY.
AND WHEN YOU'RE OUR AGE YOU DON'T EVEN DARE BLINK.
3-1
www.comics.com
picklescomic@sbcglobal.net
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Time Flies!

Discussion questions

1. Write a fraction that represents 1 year of your life if you were 13. Convert that fraction to a percent. Then write a fraction that represents 1 year of your life if you were 12 and convert that number to a percent. How much shorter would a year seem at age 13?
2. As a fraction, a percent, and in real time, how much shorter should a year seem when you are 20 than when you were 10? In real time, how much shorter should a day seem when you are 20 than when you were 10?
3. At what age would 1 year be 1/2 of your life? Explain your reasoning.
4. Find out the ages of two of your older family members. Explain to them why a year would seem to pass more quickly for the older person, using the same logic as in the cartoon. Report back to the class whether or not they understood, and their reaction to the logic.

(*Solutions on page 259*)

"Cartoon Corner" is designed to include cartoons that can be used to highlight mathematics in an interesting way. The questions accompanying the cartoons can be used as a large- or small-group activity, as homework, or as an extra-credit assignment. **Readers are encouraged to submit cartoons and classroom-tested questions to the editor, Andy Reeves, at areeves@stpt.usf.edu. The cartoons must include the date and the newspaper syndicate that holds the copyright.**

EDWARD S. MOONEY

Elizabeth's Long Walk

THE FOLLOWING "SOLVE IT" TASK APPEARED IN the October 2005 issue of this journal:

> Elizabeth visits her friend Andrew and then returns home by the same route. She always walks 2 kilometers per hour (km/h) when going uphill, 6 km/h when going downhill, and 3 km/h when on level ground. If her total walking time is 6 hours, then what is the total distance she walks in kilometers?

Denise Javaherforoush sent in solutions by her seventh- and eighth-grade students at Candlewood Middle School in Dix Hills, New York. Many of her students, like seventh-grader Devon, used a trial-and-error approach to find a solution (see **fig. 1**). In the process of solving the task, many students considered Elizabeth's walk to contain a level part, an uphill part, and a downhill part. Devon's strategy is representative of the work of many students, who concluded that each part was the same distance, 3 km. It is unclear how they came to this conclusion.

Ben, a seventh-grade student, on the other hand, decided that Elizabeth's path could be entirely level (see **fig. 2**). He assumed, like other students, that the time that Elizabeth took to travel to Andrew's house had to be the same as the time she spent returning from Andrew's house. However, Javaherforoush indicated that a few students found solutions that did not have the travel time *to* Andrew's house being the same as the travel time *from* Andrew's house. Two examples of solutions with unequal travel times to and from Andrew's house are shown in **figure 3**. (Note: The distances are not necessarily to scale.)

Samantha, an eighth-grade student in Cheryl Fantaski's mathematics class at Ruckel Middle School in Niceville, Florida, presented an algebraic way of solving the task (see **fig. 4**). In her method, K represented the total distance traveled, U denoted the total uphill distance traveled, D was the total downhill distance traveled, and L referred to the total level-ground

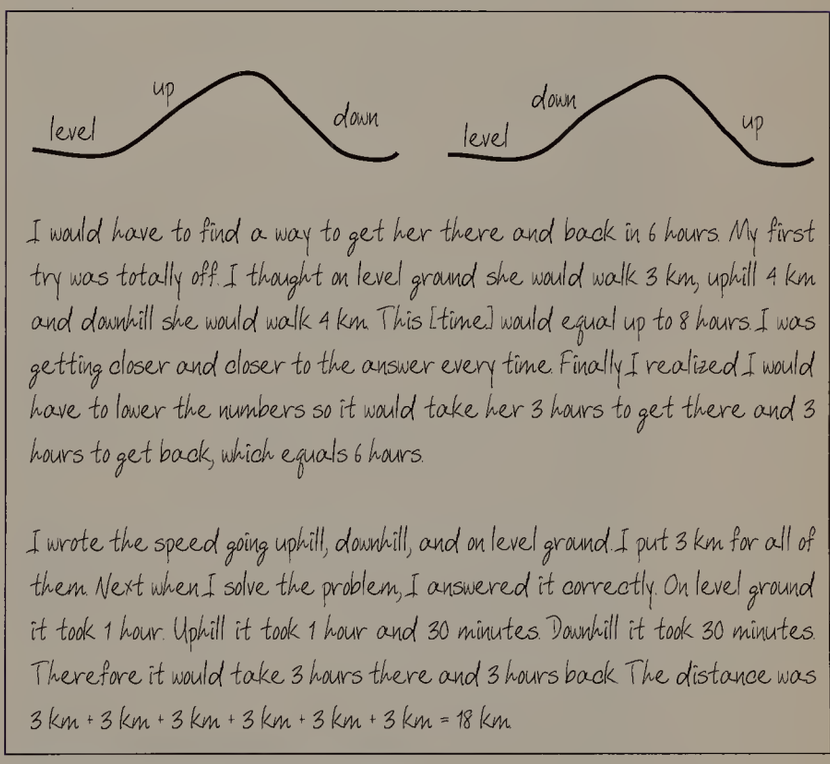
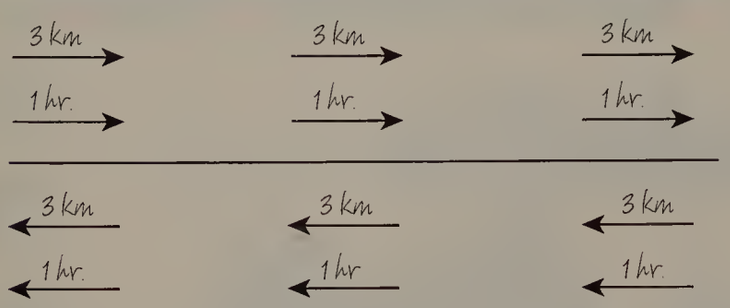

level
up
down
level
down
up
I would have to find a way to get her there and back in 6 hours. My first try was totally off. I thought on level ground she would walk 3 km, uphill 4 km and downhill she would walk 4 km. This [time] would equal up to 8 hours. I was getting closer and closer to the answer every time. Finally I realized I would have to lower the numbers so it would take her 3 hours to get there and 3 hours to get back, which equals 6 hours.
I wrote the speed going uphill, downhill, and on level ground. I put 3 km for all of them. Next when I solve the problem, I answered it correctly. On level ground it took 1 hour. Uphill it took 1 hour and 30 minutes. Downhill it took 30 minutes. Therefore it would take 3 hours there and 3 hours back. The distance was 3 km + 3 km + 3 km + 3 km + 3 km + 3 km = 18 km.

Fig. 1 Devon's trial-and-error strategy

Prepared by EDWARD MOONEY, *mooney@ilstu.edu, Illinois State University, Normal, IL 61790-4520*

This department shares the thinking of middle school students as they explore and communicate mathematics. It highlights students' work, including projects, investigations, or creative solutions to the problems in the monthly "Solve It!" department. Original student work is included along with sufficient information about the activity so that readers can try the ideas with their students. Please send manuscripts to the editor, EDWARD MOONEY, *Illinois State University, Mathematics Department, 313 Stevenson Hall, Normal, IL 61790-4520, or by e-mail to mooney@ilstu.edu. Published solutions will be credited.*



While thinking about this problem, I figured why not go for the easy approach. I figured that if it took 6 hours, then it has to take 3 hours there and 3 hours back. Because every 3 km going on flat ground is one hour, why not just go on flat ground the entire time.

Every 3 km was equal to 1 hour meaning that if I were to walk 9 km going in 3 hours, the going back would be the same. With this information it leaves the answer to be 18 kilometers.

Fig. 2 Ben's level strategy

distance that was traveled. She realized that the total is the sum of the distance traveled on level ground, uphill, and downhill ($K = L + U + D$). She also realized that the total distance traveled uphill is also the same as the total distance traveled downhill ($U = D$), since every distance uphill to Andrew's house will be the same distance downhill from Andrew's house.

The total time traveled uphill is the distance traveled uphill divided by the rate of walking, or (1 hr./2 km) • U. In a similar manner, the time traveled downhill and on level ground are (1/6) • D and (1/3) • L, respectively. The total time traveled must be 6 hours, so $6 = (1/2)U + (1/6)D + (1/3)L$. Since Samantha knew that the total distance traveled uphill is the same as the total distance traveled downhill, she substituted U for D in the equation. She then simplified the equation and solved for the total uphill distance traveled in terms of the total level-ground distance traveled, $9 - (1/2)L$, which will also be the same expression for the total downhill distance traveled, since $U = D$. Samantha then rewrote the equation for the total distance traveled, K, in terms of the total distance traveled on level ground. When the equation was simplified, she found the total distance traveled to be 18 kilometers.

Samantha's method allows us to look at solutions to the task in a more general manner. She did not rely on knowing the actual distances traveled uphill, downhill, or on level ground. If we use the equality that $U = 9 - (1/2)L = D$, as explained above, we can set up a table of various paths that would take Elizabeth 6 hours to travel (see **table 1**). The shaded areas in the table are the solutions for the paths in **figure 3** and the equidistant level-uphill-downhill path used by most students. Notice that the sum of the distances is always 18 km. **Figure 5** shows the formulas that can be used to create the table on a spreadsheet. This will allow the exploration of various solutions by placing different values for the total distance traveled on level ground.

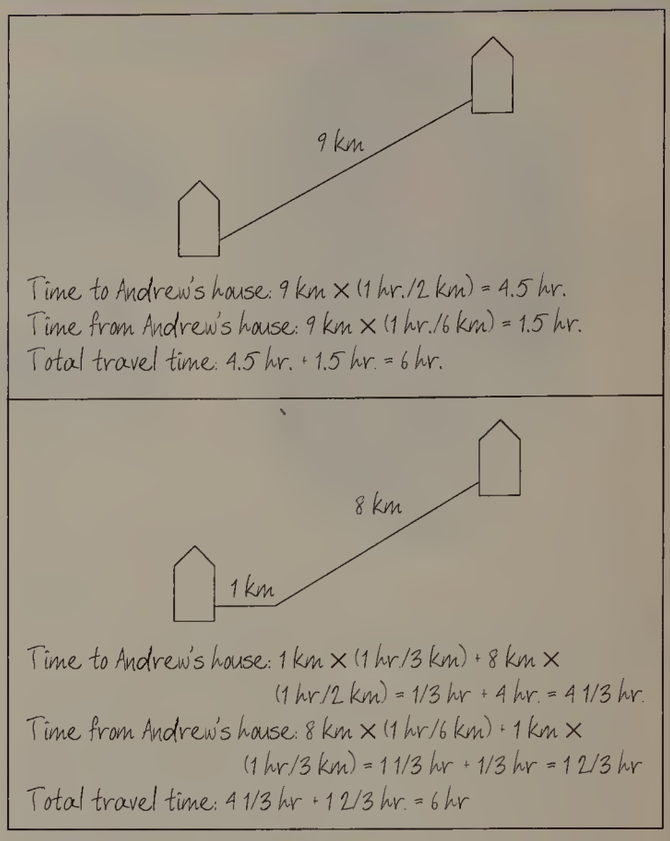

9 km
Time to Andrew's house: 9 km × (1 hr./2 km) = 4.5 hr.
Time from Andrew's house: 9 km × (1 hr./6 km) = 1.5 hr.
Total travel time: 4.5 hr. + 1.5 hr. = 6 hr.
8 km
1 km
Time to Andrew's house: 1 km × (1 hr/3 km) + 8 km × (1 hr/2 km) = 1/3 hr + 4 hr. = 4 1/3 hr.
Time from Andrew's house: 8 km × (1 hr/6 km) + 1 km × (1 hr/3 km) = 1 1/3 hr + 1/3 hr = 1 2/3 hr
Total travel time: 4 1/3 hr + 1 2/3 hr. = 6 hr

Fig. 3 Two solutions showing unequal travel times to and from Andrew's house

K represents the total distance traveled
U represents the total uphill distance traveled
D represents the total downhill distance traveled
L represents the total level distance traveled

$K = L + U + D$ $\qquad U = D$

$6 = (1/2)U + (1/6)D + (1/3)L$
$6 = (1/2)U + (1/6)U + (1/3)L$
$6 = (2/3)U + (1/3)L$

$U = 9 - (1/2)L$

$D = U = 9 - (1/2)L$

$K = L + U + D$
$K = L + [9 - (1/2)L] + [9 - (1/2)L]$
$K = 18$

The total distance is 18 kilometers.

Fig. 4 Samantha's algebraic strategy

TABLE 1

Different Possibilities for Level, Uphill, and Downhill Distances with a 6-Hour Travel Limit

Total Level Distance (km)	Total Uphill Distance (km)	Total Downhill Distance (km)	Level Time (hr.)	Uphill Time (hr.)	Downhill Time (hr.)	Total Time (hr.)
0	9	9	0	4 1/2	1 1/2	6
1	8 1/2	8 1/2	1/3	4 1/4	1 5/12	6
2	8	8	2/3	4	1 1/3	6
3	7 1/2	7 1/2	1	3 3/4	1 1/4	6
4	7	7	1 1/3	3 1/2	1 1/6	6
5	6 1/2	6 1/2	1 2/3	3 1/4	1 1/12	6
6	6	6	2	3	1	6
7	5 1/2	5 1/2	2 1/3	2 3/4	11/12	6
8	5	5	2 2/3	2 1/2	5/6	6
9	4 1/2	4 1/2	3	2 1/4	3/4	6
10	4	4	3 1/3	2	2/3	6
11	3 1/2	3 1/2	3 2/3	1 3/4	7/12	6
12	3	3	4	1 1/2	1/2	6

	A	B	C	D	E	F	G
1	Level distance (km)	Uphill distance (km)	Downhill distance (km)	Level time (hr.)	Uphill time (hr.)	Downhill time (hr.)	Total time (hr.)
2	A2	= 9 – 1/2 * A2	= B2	= (1/3) * A2	= (1/2) * B2	= (1/6) * C2	= D2 + E2 + F2

Fig. 5 Spreadsheet formulas used to construct table. (Note: "A2" indicates the value in column A row 2 of the spreadsheet.–*Ed.*)

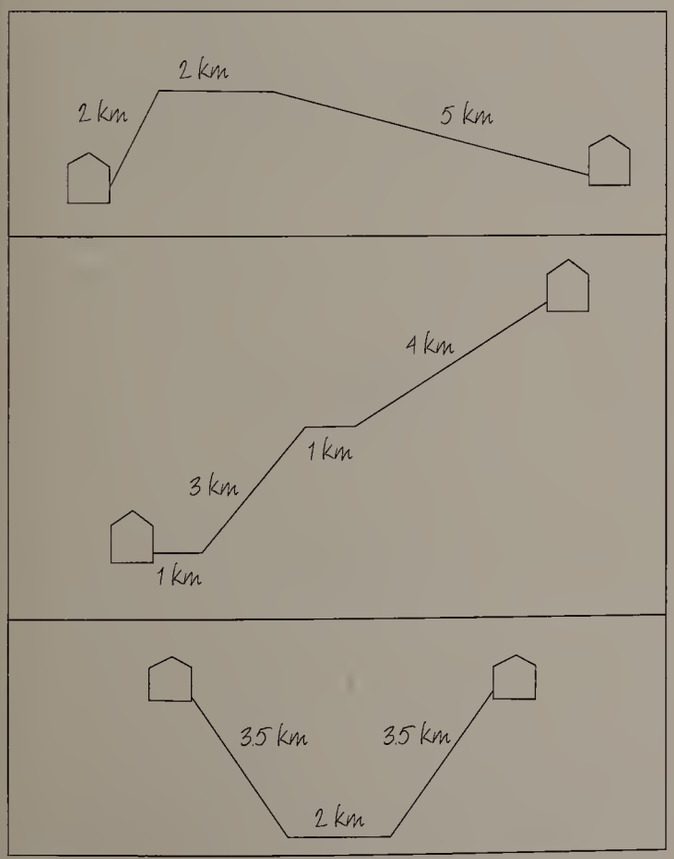


Fig. 6 Paths with total level, uphill, and downhill travel distances of 4 km, 7 km, and 7 km, respectively

In Elizabeth's walk, the total level-ground distance traveled is twice the distance of the level ground for the path. Elizabeth will always walk the level-ground distance twice, going to Andrew's house and then back home. However, the total uphill distance is based on the sum of the distance traveled uphill going to Andrew's and the distance traveled uphill going back. These distances are not necessarily the same. The same idea holds true for the downhill distance. **Figure 6** shows three paths that have total level, uphill, and downhill travel distances of 4 km, 7 km, and 7 km, respectively. Notice that since the table deals with the total distance, the uphill part of the path, for example, does not have to be one single part of the path, as seen in the second example in **figure 6**.

The Editorial Panel congratulates these students for their solutions to the problem and encourages more teachers to submit the work of their students, which shows their mathematical thinking and reasoning. We would like to thank the following teachers for submitting the work of their students:

Cheryl Fantaski, Ruckel Middle School, Niceville, Florida

Denise Javaherforoush, Candlewood Middle School, Dix Hills, New York □

LAURA J. MOSS AND BARBARA W. GROVER

Not Just for Computation:

Basic Calculators Can Advance the Process Standards

THE NATIONAL COUNCIL OF TEACHERS OF Mathematics (NCTM), the International Society for Technology in Education (ISTE), and the Conference Board of the Mathematical Sciences (CBMS) are among the many organizations that have advocated the use of various forms of technology (e.g., computer programs, graphing/scientific/four-function calculators) to educate students. A major reason for this advocacy is captured well in the National Technology Standards comments on using technology in mathematics learning activities. "Technology can play a role in enhancing mathematical thinking, student and teacher discourse, and higher-order thinking by providing the tools for exploring and discovering mathematics" (ISTE 2000, p. 96).

These three components of the role that technology can play are embedded in the five Process Standards (NCTM 2000) and in the Recommendations for Technology in Teacher Preparation (CBMS 2001). Graphing calculators and computer programs are often the technological tools that immediately come to mind when reading such a statement. Four-function calculators and the TI-Explorer can play the same role. (Texas Instruments has recently combined the TI-Explorer and the TI-73 graphing calculators into one device. The activities in this article relate to the TI-Explorer calculator. Although a TI-73/Explorer can be used, the keystrokes may be slightly different from those identified here.) These calculators are often more prevalent in schools around the country. This article addresses ways in which to use nongraphing calculators to enhance multiple problem-solving strategies, encourage discourse, and facilitate high-level mathematics thinking. In addition, sample activities allow students to become familiar with multiple ways of using a calculator to obtain an answer, learn to interpret the calculator display, and identify the limitations of the calculator. The two examples described in detail are purposefully simple situations. Their simplicity fosters a positive "I-can-do-this" attitude, encouraging students to contribute to discussions. The focus is on developing student abilities to think and reason. The additional examples listed are more complicated at the outset. This instructional approach can be used at many grade levels.

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The instructional recommendations are variations on one basic strategy. After students individually estimate a solution, the following work can be done in pairs or small groups before sharing with the entire class.

- Present a problem in a real-world context.
- Require students to individually record a solution estimate in ink and justify their estimate.
- Ask students to share and discuss estimates and rationales. They should do this first in pairs, before whole-class discussion. Students comment on the extent to which they are convinced of the accuracy of the thinking. The teacher remains neutral in word and body language to all contributions.
- Have students use a calculator to solve the problem, keeping track of the keystrokes they use so they can share their work with their classmates.
- Ask students to discuss the different approaches and identify similarities and differences in using the calculator.
- Interpret the solution displayed on the calculator.

We now consider two examples to illustrate this instructional strategy. The first example is a percent problem; the second example, a basic word problem, addresses a division situation in which the quotient is not a whole number.

Percent Problem

In our school, 25% of the 42 eighth graders walk to school. How many eighth-grade students walk to school? (Problem adapted from Bitter and Mikesell 1990, pp. 354–57)

At first glance, this simple, straightforward problem emphasizes a computation procedure. Students enter .25 × 42, obtain an answer, and are done. However, this problem can be rich and engaging and can focus on important mathematics identified in the Process Standards, as shown in the discussion that follows.

Individually record a solution estimate in ink: Every student should have the **Percent Problem Worksheet**, pen, and pencil as well as a calculator. Each student is instructed to silently read the problem and *in ink* write an estimate of the answer along with a rationale. Using ink reduces the temptation to change an estimate and/or rationale.

As a whole class, discuss estimates and rationales: The discussion of these estimates should be conducted by displaying the procedures on an overhead projector or on the chalkboard while listening to the students' reasons. The discussion facilitator need not comment on the accuracy of any of the estimates or rationales but should seek student opinions about their peers' contributions. Some students might estimate a solution using their knowledge of 10 percent of a number, resulting in an estimate of 8, 10, or 12. Others may use the fact that 25 percent is equivalent to 1/4 and use the fraction to obtain an estimate of 10. It is possible that some students may believe that 25 percent represents 25 students and use that as their estimate. A student could also ignore the mathematics and find an estimate based on the number of students they know walk to school. The facilitator records each of these thoughts so that the initial ideas remain public throughout the time required to work the problem. A discussion of these estimates as well as their rationales will encourage discourse, help students to develop the ability to take risks and rely on their own thinking, expose students to multiple ways of thinking about a problem, emphasize criticism of the thinking not the person, and motivate students to complete the next task (actual computation) to determine "Am I right?"

Individually use a calculator to solve: Students use their calculators to obtain a solution, noting each keystroke on their worksheets. Some students may enter 25, %, ×, 42, =. Others might enter .25, ×, 42, =. Someone may enter 42, ×, 25, %, =. Some students might enter keystrokes that yield an incorrect response.

As a whole class, discuss the different approaches to using the calculator: Questions to stimulate discussion might include these: Did everyone obtain the same result? Why? Which approach, if any, minimizes keystrokes and could be considered more efficient? By discussing different approaches and analyzing why answers are the same or different, focus students' attention on underlying mathematics concepts including decimal/percent equivalents, the meaning of various operations, and the commutative property of multiplication.

Interpret the result displayed on the calculator: Finally, the discussion should turn to the answer, 10.5. What does this number shown in the display mean? A simple interpretation is that 25 percent of 42 is 10.5. However, in the context of the original problem, how should the students interpret the value to answer the question How many students walk to school? One-half a student is not an acceptable answer; it should obviously be a whole number because we are referring to people. Thus, a discussion may now focus on interpreting this value. Should the correct answer be 10 or 11 students? Why choose one answer over the other? This discussion will allow students to bring in their personal experiences and views and may help them

develop analytic as well as critical-thinking skills. Every student may contribute to this dialogue, not just those who accurately calculated the result. Both 10 and 11 could be considered correct responses as long as the student is able to adequately justify the choice.

Now let us consider a situation that involves interpreting noninteger quotients and two ways in which calculators may present those results.

Basic Word Problem

> Robb wants to read 100 pages of his book before his next conference with his teacher in 7 days. How many pages should he read each day? (Problem adapted from CBMS 2001b, p. 71)

The next two steps, individually record a solution estimate in ink and discuss estimates and rationales, follow the same pattern as the percent problem (see the **Reading a Book Problem Worksheet**).

Individually record a solution estimate in ink: In this example, if an Explorer calculator is available, the teacher may instruct *all* students to use the INT ÷ feature or instruct some to use the INT ÷ feature and others to use the standard division (÷) key. Students record their keystrokes. The INT ÷ feature displays quotients as integer values with remainders. In this example, 100 INT ÷ 7 will be displayed as 14 with a remainder of 2. The standard division key displays quotients as decimals. In this example, 100 ÷ 7 is displayed as 14.285714 (or as many decimal places as appropriate for a particular calculator).

As a whole class, discuss the different approaches to using the calculator. As many of the readers will be aware, students are likely to enter the values of 100 and 7 in an incorrect order. A discussion of the different approaches should include a focus on the lack of a commutative property for division (e.g., 100 INT ÷ 7 ≠ 7 INT ÷ 100). If some of the students used the standard division key, then additional discussion can ensue with respect to the difference in answers displayed. What does the calculator do differently in each case? How could they imitate the calculator's computation in the two situations with paper and pencil?

Interpret the result displayed on the calculator. What is the meaning of 14 r2 in the one case and 14.285714 in the other? What does the 14 represent? (He must read 14 pages per day.) What does the 2 represent? (If Robb reads exactly 14 pages per day for 7 days, he will still have 2 pages to read.) What would the students suggest Robb do to finish the assignment on time? Many "correct" answers are possible to this question (e.g., 16 pages the first day, then 14 every day thereafter; 16 pages on the last day; 15 pages on 2 of the 14 days, and 14 on the rest). The discussion should focus on reasoning, thinking, and justifying solutions. The interpretation of 14.284714 provides different challenges. The 14 means the same as it did in the 14 r2 display, but the 0.284714 means something quite different from the r2. It means 0.284714 *of a page.* The discussion gives students an opportunity to deepen their understanding of decimals and allows teachers to identify students who have a good understanding of decimals and those who need more help. Does it make sense for Robb to try to read 14.284714 pages per day? What are the difficulties involved? What are the advantages and disadvantages of the two formats of the answer?

On the **Reading a Book Problem Worksheet**, students are asked to indicate the difference between their estimate and the actual solution. This value can be tracked by the teacher as a way to assess an individual student's progress in estimation skills. The names of the top five students who *improve* their ability to estimate each month could be posted as well as the five students who have the best *average* difference between their estimates and the actual answers. We suggest the number 5 because it gives several students the opportunity to have his or her name on a list, not just one best student, and can be an incentive to improve performance. Teachers can adjust this number for their individual situations. Another approach would be to have individuals set goals for themselves and track their progress toward their goals.

Additional Examples

THESE TWO EXAMPLES PRESENTED SIMPLE, straightforward situations. Textbooks include similar situations that can be used to encourage deep investigation and foster communication about mathematics concepts among the students. The teacher gains significant insights into the thinking and reasoning of students, an understanding of students' proficiency or lack thereof in using a calculator appropriately, and a more thorough ongoing assessment of students' abilities. Some additional examples of real-world contexts (and their solutions) are given below.

> 1. Three pizzas need to be shared among 17 people. If the pizza is to be equally divided, how much pizza does each person get? (Replace the 3 pizzas with 3 dollars or 3 dozen doughnuts for alternative situations.)

3 pizzas

Using the integer division key (INT ÷): The display will be 0 r3. This display is interpreted as no one (0 people) can have a whole pizza and that 3 whole pizzas remain to be shared among the 17 people. Each person's share is 3/17 of a pizza.

Using the standard division (÷) key: Each person's share is displayed as 0.1764706 of a pizza. This exact decimal value is useless in this real-life problem. This situation should generate discussion about the difficulty of actually carrying out division for either of these approaches and could include what is more likely in a real-world setting. The pizzas will get cut into reasonably equal pieces, usually 8, and some people will eat more than others.

3 dollars

Using 3 divided by 17 will result in the same calculator display as for the pizzas. The INT key results in 0 r3; each person gets 0 dollars and the remaining $3 must be shared among 17. Each person's share is 3/17 of a dollar, or 0.1764706 of a dollar. These values are difficult to imagine. Since a dollar bill cannot be sliced like a pizza, it is useful to convert the $3 to 300 cents.

INT ÷ key: 300 INT ÷ 17 results in 17 r11. Each person gets 17 cents, and 11 cents are left over. Discussion ensues about what to do with the 11 cents. Again, since the remainder cannot be sliced and equally divided as can the pizza, answers are "give it to charity" or distribute funds unevenly. Equal distribution of all the money is impossible.

Standard ÷ key: Using 300 ÷ 17 results in 17.647059. The 17 again means that each person's share is 17 cents. The 0.647059 means each person's share in addition to the 17 cents is a bit more than half a penny (0.5 is half a penny and 0.6 > 0.5). Again, this portion cannot be distributed.

3 dozen doughnuts

The 0 r3 means that no one person can have a full dozen doughnuts; the 3 dozen must be shared among the 17 people. Each person's share is 3/17 of a dozen or 0.11764706 of a dozen. It is useful to convert the 3 dozen doughnuts to 36 doughnuts to help give meaning to these values.

INT ÷ key: 36 INT ÷ 17 results in 2 r2; each person's share is 2 doughnuts with 2 doughnuts left over, or 2 doughnuts and 2/17 of a doughnut. The 2 doughnuts could each be divided into 17 equal pieces, and each person will get 2 of the 1/17 pieces.

Standard ÷ key: 36 ÷ 17 results in 2.1176471; each person's share is 2 doughnuts and 0.1176471 of a doughnut. This fractional part for each person is about 0.1 of a doughnut. As with the pizza, in a real-world setting two people would just eat the extra doughnuts.

2. Repeat problem 1, but divide the pizzas, dollars, or doughnuts among 15 people and then 2 people. What different mathematics comes into play?

3 pizzas for 15 people

INT ÷ key: 3 INT ÷ 15 = 0 r3. Each person's share is 0 whole pizzas, and the remaining 3 pizzas must be divided. Each person's share is 3/15, or 1/5, of a pizza, interpreted as "divide each pizza into 5 slices and everyone takes a slice."

Standard ÷ key: 3 ÷ 15 = 0.2; each person's share is 0.2 of a pizza, or 1/5.

3 dollars for 15 people

INT ÷ key: 3 INT ÷ 15 = 0 r3; each person's share is 0 dollars and the remaining 3 dollars must be divided. Each person gets 3/15, or 1/5 of a dollar, which is 20 cents.

Standard ÷ key: 3 ÷ 15 = 0.2; each person's share is 0.2 of a dollar, or 20 cents. It is not necessary to convert the $3 to 300 cents to make sense of the amount that is each person's share, however, it is a worthwhile exercise to see that the solutions are equal.

3 dozen doughnuts for 15 people

INT ÷ key: 3 INT ÷ 15 = 0 r3; each person's share is 0 dozen doughnuts. The remaining 3 dozen must be divided among the 15 people. Each person's share is 3/15, or 1/5 of a dozen; 1/5 of 12 = 2 2/5, or 2 doughnuts and 2/5 of another doughnut.

Standard ÷ key: 3 ÷ 15 = 0.2; each person's share is 0.2 of a dozen doughnuts; 0.2 of 12 = 2.4, or 2 doughnuts and 0.4 of another doughnut. It is not necessary to convert the 3 dozen to 36 doughnuts, but it is worthwhile to investigate.

3 pizzas for 2 people

INT ÷ key: 3 INT ÷ 2 = 1 r1; each person's share is 1 whole pizza. The remaining pizza must be divided between the 2 people. Each person's share is 1 whole pizza and 1/2 of the other pizza.

Standard ÷ key: 3 ÷ 2 = 1.5; each person's share is 1 whole pizza and 0.5 of a pizza.

3 dollars for 2 people

INT ÷ key: 3 INT ÷ 2 = 1 r1; each person's share is 1 whole dollar. The remaining dollar must be divided by

2. Each person's share is 1 dollar and 1/2 of a dollar.

Standard ÷ key: 3 ÷ 2 = 1.5; each person's share is 1 dollar and 0.5 of a dollar or a total of $1.50. Again, it is not necessary to convert the $3 to 300, but it shows that the solutions have equal value although the numbers (1.5 and 150) and the labels (dollars or cents) are different.

3 dozen doughnuts for 2 people

INT ÷ key: 3 INT ÷ 2 = 1 r1; each person's share is 1 dozen doughnuts and the remaining dozen must be divided between the 2 people. Each person's share is 1 dozen and 1/2 dozen.

Standard ÷ key: 3 ÷ 2 = 1.5; each person's share is 1.5 dozen doughnuts. The conversion to 36 doughnuts is not necessary, but again it is worth investigating.

3. Ask each student, "How many seconds old are you?" given a specific ending time.

Discuss issues regarding various units of time. The class needs to decide on guidelines so that all students are using the same criteria for determining the solution. If years are used, will leap years be considered or disregarded? If months are used, will every month have 30 days, should the actual number of days of each month be used, or will every month have 4 weeks (28 days)?

Suppose a student was born June 15, 1994, at 7:15 p.m. The ending time established by the class or teacher is October 20, 2005, 1:22 p.m. Count leap years, and specify the days in a month. One approach: From June 15, 1994, 7:15 p.m. to June 15, 2005, 7:15 p.m. is 11 years, with leap years in 1996 and 2004. The number of days from June 15, 2005, 7:15 p.m. to October 19, 2005, 7:15 p.m. is 126 days. June has 15 days (from June 16 to 30); July, 31 days; August, 31; September, 30; and October, 19. The number of hours and minutes from October 19, 2005, 7:15 p.m. to October 20, 1:22 p.m. is 18 hours, 7 minutes. This student has been alive 11 years, 128 days (include the 2 leap year days), 18 hours, and 7 minutes. With 60 seconds in 1 minute, 60 minutes in 1 hour, 24 hours in 1 day, 365 days in 1 year, we have 31,536,000 seconds in 1 year, 86,400 seconds in 1 day, and 3,600 seconds in 1 hour. The number of seconds this student has been alive is

$$\begin{aligned} &31{,}536{,}000 \text{ sec./yr.} \times 11 \text{ yr.} + 86{,}400 \text{ sec./day} \\ &\quad \times 128 \text{ days} + 3{,}600 \text{ sec./hr.} \times 18 \text{ hr.} \\ &\quad + 60 \text{ sec./min.} \times 7 \text{ min.} \\ &= 346{,}896{,}000 \text{ sec.} + 11{,}059{,}200 \text{ sec.} \\ &\quad + 64{,}800 \text{ sec.} + 420 \text{ sec.} \\ &= 358{,}020{,}420 \text{ sec.} \end{aligned}$$

Depending on the calculator and the age of the students, the values may be displayed in scientific notation or as a decimal with an exponent. Values may get rounded along the way. Such situations raise important issues about the limitations of some calculators and the need for understanding decimals, powers of 10, and scientific notation.

4. If you could walk a million strides, how far would you walk in miles? In kilometers? Suppose you walked 3 miles per hour (mph) and could walk 8 hours a day, how long would it take you to walk 1 million strides?

A teacher can allow the students to determine the length of their stride before starting the calculations or give the students an average measure of a stride for students at their grade level. For this solution, a stride is considered to be 18 in., or 45 cm. This activity can motivate a discussion on the advantages of the metric system.

In miles:

$$1{,}000{,}000 \text{ strides} \approx 18{,}000{,}000 \text{ in.} \div \frac{63{,}360 \text{ in.}}{1 \text{ mi.}}$$

$$= 18{,}000{,}000 \text{ in.} \cdot \frac{1 \text{ mi.}}{63{,}360 \text{ in.}} \approx 284.09 \text{ mi.}$$

In kilometers:

$$1{,}000{,}000 \text{ strides} \approx 45{,}000{,}000 \text{ cm} \div \frac{100{,}000 \text{ cm}}{1 \text{ km}}$$

$$= 45{,}000{,}000 \text{ cm} \cdot \frac{1 \text{ km}}{100{,}000 \text{ cm}} = 450 \text{ km}$$

In time:

$$\frac{3 \text{ mi.}}{1 \text{ hr.}} \cdot \frac{8 \text{ hr.}}{1 \text{ day}} = \frac{24 \text{ mi.}}{1 \text{ day}}$$

$$284.09 \text{ mi.} \div \frac{24 \text{ mi.}}{1 \text{ day}} \approx 11.837 \text{ days}$$

$$\frac{3 \text{ mi.}}{\text{hr.}} \approx \frac{4.83 \text{ km}}{\text{hr.}}$$

$$\frac{4.83 \text{ km}}{\text{hr.}} \cdot \frac{8 \text{ hr.}}{\text{day}} = \frac{38.64 \text{ km}}{\text{day}}$$

$$450 \text{ km} \div \frac{38.64 \text{ km}}{\text{day}} = 450 \text{ km} \cdot \frac{1 \text{ day}}{38.64 \text{ km}} \approx 11.65 \text{ days}$$

Why are the values in the standard system different from those in metric?

5. If the federal debt is 7.92 trillion dollars, how could you express this value in terms that an average citizen would understand? What denomination of bill would you need to use to lay a path from the earth to the moon that would use all 7.92 trillion dollars?

This problem addresses issues related to limitations of a calculator, powers of 10, scientific notation, and placement of a decimal point. The national debt value is based on the U.S. Treasury Bureau of Public Debt Web site (www.publicdebt.treas.gov/opd/opdpenny.htm), which reported the number as $7,922,079,988,192 on September 26, 2005. The estimated population can be obtained from the U.S. Census Bureau Web site (www.census.gov/population/www/popclockus.html). The estimated population on September 27, 2005, was 297,278,263. The teacher should go to these two Web sites to obtain accurate values close to the date of their use since the debt and the population increase daily.

Average citizen understanding: Students may suggest multiple approaches that address average citizen understanding. This solution uses the approach of how much each citizen would need to pay to erase the debt.

Most nongraphing calculators display a maximum of eight digits. Consequently, discussions need to occur that address approaches to dealing with these large numbers and the impact these choices have on the level of accuracy of the results. We suggest converting to scientific notation with three significant digits to maintain a consistent level of accuracy. The national debt is then expressed as 7.92×10^{12}. The population is expressed as 2.97×10^{8}. The calculator display for $7.92 \div 2.97$ is 2.6666666. Students can mentally divide 10^{12} by 10^{8}. The final result is 2.6666666×10^{4}, or 26,666.666. Each U.S. citizen's portion of the national debt is approximately $26,700.00.

Path from the earth to the moon: The mean distance from the earth to the moon is 384,403 kilometers. All U.S. paper currency is the same size, 15.5 cm by 6.5 cm. We assume the bills will be placed end to end. We use scientific notation again.

Distance to the moon: 3.84×10^{5} km = 3.84×10^{10} cm because 1 km = 1,000 m = 100,000 cm = 10^{5} cm. The length of U.S. paper currency is 15.5 cm = 1.55×10^{1} cm. The number of bills needed: $(3.84 \times 10^{10}\text{cm}) \div (1.55 \times 10^{1}\text{cm/bill}) = 2.48 \times 10^{9}$ bills. The value of each bill on the path needed to equal the national debt: $(7.92 \times 10^{12}$ dollars$) \div (2.48 \times 10^{9}$ bills$) = 3.19 \times 10^{3}$ dollars/bill = $3,190/bill. U.S. currency does not include $3,000 bills. Thus, someone could lay $1,000 bills from the earth to the moon, back to the earth, and back to the moon again and still have some extra bills. Many different solutions are possible with respect to the denomination of the bill used. These two results can give students (and their parents) a better sense of the meaning of our national debt.

Conclusion

SIMPLE NONGRAPHING CALCULATORS CAN BE powerful tools to enhance students' conceptual understanding of mathematics concepts. Students have opportunities to develop (a) a broad repertoire of problem-solving strategies by observing multiple solution strategies; (b) respect for other students' abilities and ways of thinking about mathematics; (c) the ability to critically analyze mathematical ideas, not people; (d) the ability to collaborate and cooperate with diverse personalities; (e) confidence and competence in doing mathematics; and (f) the ability to persevere even when a task is difficult.

Communities of learners at two levels can be fostered: small groups and whole class. Students work in pairs or small groups establishing small learning communities with a subset of their peers, and the entire class becomes a learning community as the teacher builds a supportive and nurturing environment in which students and the teacher learn from one another.

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(See worksheets on the next pages.) □

Percent Problem Worksheet

NAME _____

In our school, 25% of the 42 eighth graders walk to school. How many eighth-grade students walk to school?

1. Estimate the solution (in ink): _____ Explain why you selected this estimate.

2. Use your calculator to solve the problem. Write down all your keystrokes and which values appear in the display each time. (You may not need to fill all the spaces.)

KEYSTROKES	DISPLAY

3. *After* our class discussion, complete these items. My approach to using the calculator was (circle one):
Correct Incorrect

If correct, I could have been more efficient (used fewer keystrokes): Yes No
Support your answer.

If incorrect, the keystrokes I could have used were these:

4. Describe the meaning of the answer that the calculator displayed. Explain the meaning of the whole number part and the meaning of the remainder or the decimal in terms of students who walk to school.

5. Is the final answer displayed by the calculator the answer to the question asked in the problem? Explain.

6. The difference between my estimate and the final solution is _____.

7. One thing I learned from today's discussions is this:

From the December 2006/January 2007 issue of **Mathematics Teaching in the Middle School**

Reading a Book Problem Worksheet

NAME _____

Robb wants to read 100 pages of his book before his next conference with his teacher in 7 days. How many pages should he read each day?

1. Estimate the solution (in ink):_____ Explain why you selected this estimate.

2. Use the INT ÷ (or the ÷) feature on your calculator to solve the problem. Write down all your keystrokes and which values appear in the display each time. (You may not need to fill all the spaces.)

Keystrokes (INT ÷)	Display

Keystrokes (÷)	Display

3. After our class discussion, complete these items. My approach to using the calculator was (circle one):
   Correct Incorrect

   If correct, I could have been more efficient (used fewer keystrokes): Yes No
   Support your answer.

   If incorrect, the keystrokes I could have used were these:

4. Describe what the final answer displayed in the calculator means. Explain the meaning of the whole number part and the meaning of the remainder or the decimal in terms of pages to be read.

   For the INT ÷ procedure:

   For the ÷ procedure:

5. Is the final answer displayed by the calculator the answer to the question asked in the problem? Explain.

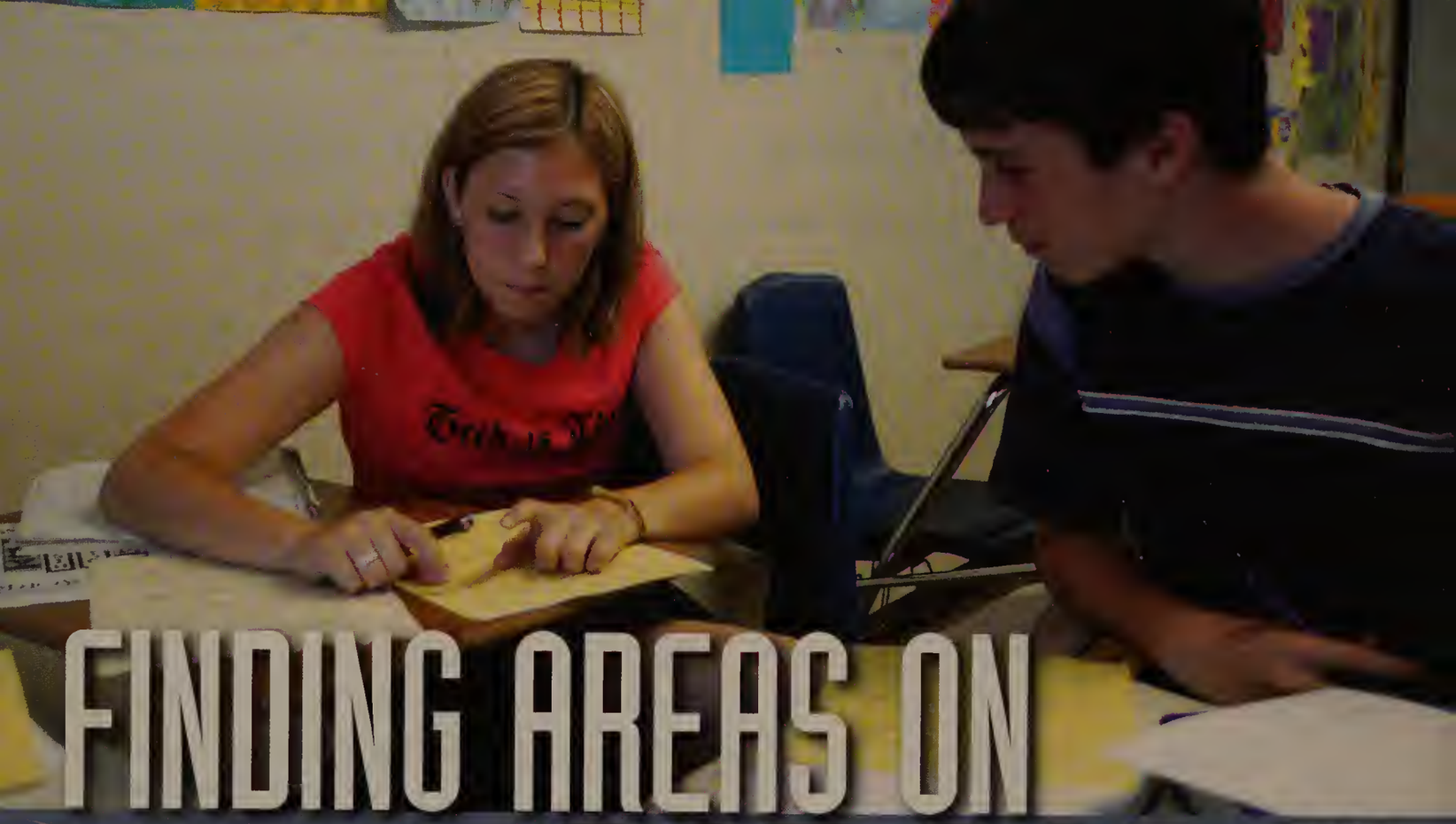
   For the INT ÷ procedure:

   For the ÷ procedure:

6. The difference between my estimate and the final solution is _____.

7. One thing I learned from today's discussions is this:

From the December 2006/January 2007 issue of **Mathematics Teaching in the Middle School**



FINDING AREAS ON DOT PAPER

DAVID L. PAGNI

FINDING THE AREAS OF RECTANGLES, TRIangles, parallelograms, and trapezoids through formulas are part of most state mathematics standards and are included in middle school mathematics textbooks. Students are expected to know these formulas and apply them to find the areas of figures shown in textbooks. But as the TIMSS 1995 video study pointed out, students in the United States are seldom given the opportunity to discover the formulas for themselves (Stigler and Hiebert 2004). This article provides activities that prompt students to investigate and discover the formulas, with a teacher's guidance. Finally, students work with a nonstandard unit of area measure, an equilateral triangle, to investigate area formulas using this unit.

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As the Connection Standard for Grades 6–8 states: "Thinking mathematically involves looking for connections, and making connections builds mathematical understanding" (NCTM 2000, p. 274).

The investigation is fun, and it gives students the opportunity to engage in, what is for them, original mathematics research. The activity also demonstrates that our standard square unit of area is somewhat arbitrary and that other units of area (e.g., the equilateral triangle) are equally viable and can yield formulas for familiar figures like squares, rectangles, triangles, parallelograms, and trapezoids. This activity has been presented in eighth-grade algebra classes and requires a class period to get started and homework to finish. Discussion the following day allows students to share their discoveries and for the teacher to discuss the connections between the square and triangular units of area measure.

Many textbooks lead into the standard formulas for finding the areas of rectangles, parallelograms, triangles, and trapezoids by using square grid paper or square dot paper. At first, students draw a rectangle and count the squares; later, they multiply length by width ($l \times w$) (see **fig. 1**). Of course, the unit

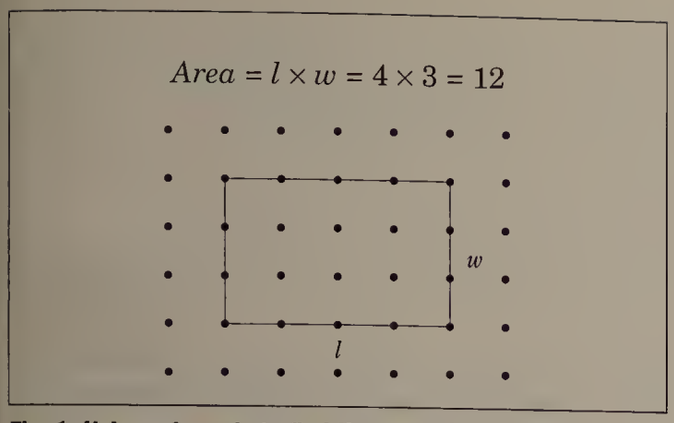


Fig. 1 Using a formula to find the area of a rectangle

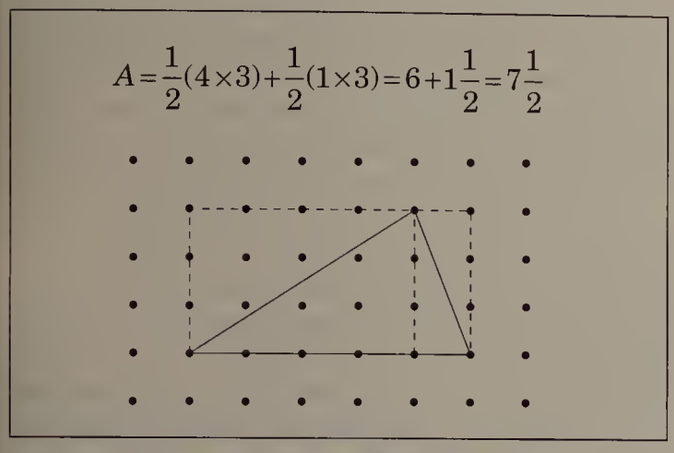


Fig. 2 Finding the area of a triangle by partitioning

of area is the 1 × 1 square on the dot paper, so in this case the area is 12 squares, or 12 square units.

If we draw a triangle on square dot paper, we can find its area by partitioning the triangle or by using the formula $A = (1/2)bh$ (b = base length, h = height). The reasoning behind the partitioning method is that a diagonal of a rectangle partitions the rectangle into two right triangles, each of which has area that is half that of the original rectangle (see **fig. 2**). Using the formula for a triangle produces the same result: $A = (1/2)bh = (1/2)(5)(3) = 15/2 = 7\ 1/2$. Of course, if we apply the distributive property to the partitioning method, we can derive the formula:

$$A=\frac{1}{2}(4\times3)+\frac{1}{2}(1\times3)=\frac{1}{2}(12)+\frac{1}{2}(3)$$

$$=\frac{1}{2}(12+3)=\frac{1}{2}(15)=\frac{1}{2}(5\times3)$$

$$=\frac{1}{2}(b\times h)$$

By constructing a rectangle that has the same area as a given parallelogram, we can use similar partitioning and reasoning to derive the formula for a parallelogram (see **fig. 3**). Also, we can construct a parallelogram from a given trapezoid and use the formula for area of a parallelogram and algebra to derive the

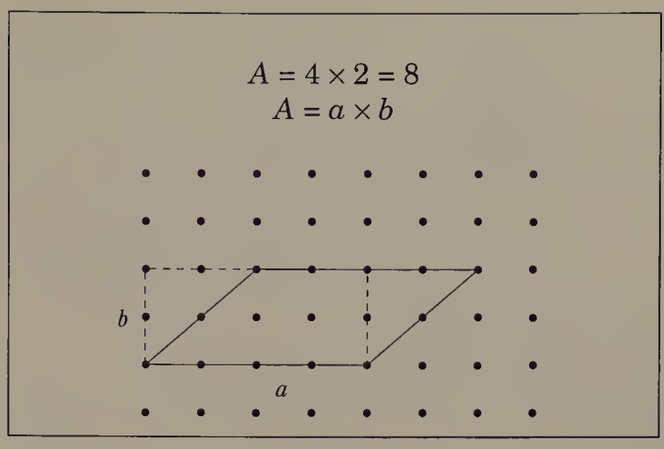


Fig. 3 Deriving the formula for the area of a parallelogram

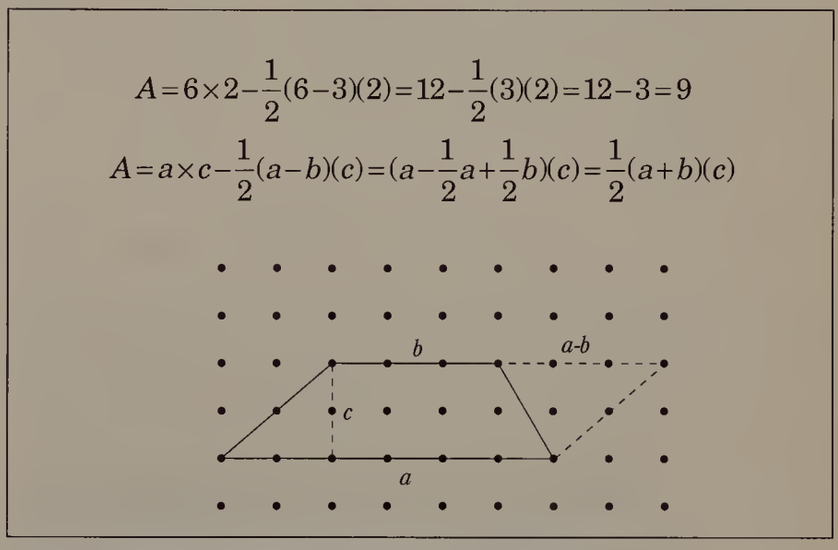


Fig. 4 Deriving the formula for the area of a trapezoid

formula for the area of a trapezoid (see **fig. 4**).

Students investigate finding areas and deriving formulas for rectangles, triangles, parallelograms, and trapezoids on square dot paper using **Activity Sheet 1** (found at the end of this article). The Geometer's Sketchpad may also be used to demonstrate the activities.

Suppose that we want to find areas on triangle dot paper (also called isometric dot paper). It is hoped that this experience will help students expand their understanding of the concepts of area and unit of area, thus giving them the opportunity to invent workable formulas of their own. In this case, the standard formulas we found on square dot paper will give us answers for area in square units, not (equilateral) triangular units. The question we ask ourselves is this: "Can we find the areas of figures using the equilateral triangle as the unit of area?" We will assume that the side of the equilateral triangle on the dot paper has the same length as the side of the square on the square dot paper.

First, we will find the area of a rectangle (in triangular units) by using the partitioning method. On the triangle dot paper, two triangles sharing a common side form a parallelogram (also a rhombus) of area 2; two of these form a parallelogram of area 4. We can make

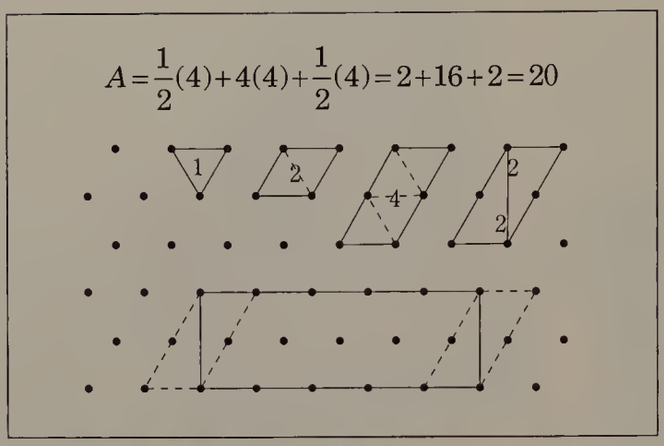


Fig. 5 Finding the area of a rectangle in equilateral triangular units

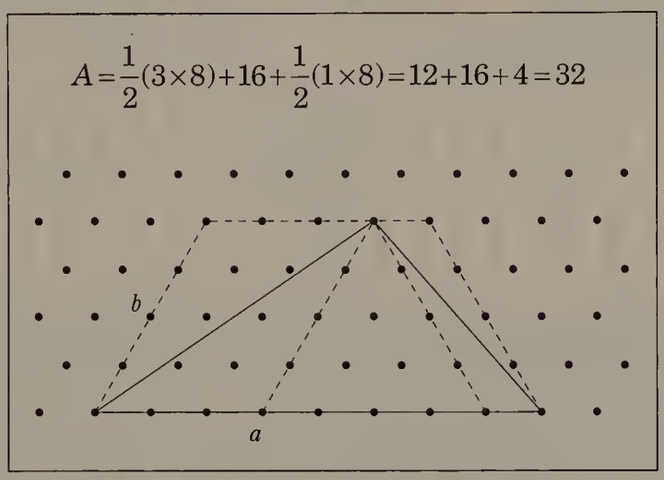


Fig. 6 Finding the area of a triangle in equilateral triangular units

larger parallelograms, the areas of which are easy to find by counting small equilateral triangles or small parallelograms. To find the area of our rectangle, we will need to partition it into small parallelograms (see **fig. 5**). Thus, the rectangle is made up of four small parallelograms each of area 4, which are congruent to the dotted parallelograms in **figure 5**. They account for 4(4) = 16 triangular units. The dotted parallelograms at the sides of the rectangle are half inside the rectangle, so they each account for (1/2) (4) = 2 triangular units.

Next, we will find the area of a triangle (in triangular units) also by using the partitioning method (see **fig. 6**). Here we partition our triangle into two "half" parallelograms, with an equilateral triangle between them (see the dotted lines). The left parallelogram is composed of three tall parallelograms with base of length 1, each comprising 8 small triangular units. Thus, since the left parallelogram is half inside the triangle of **figure 6**, it accounts for (1/2) (3 × 8) = 12 triangular units. In a similar way, the dotted parallelogram on the right with base of length 1 is also half inside the triangle of **figure 6** and accounts for (1/2) (1 × 8) = 4 triangular units. Finally, by counting one at a time from the top, the dotted equilateral triangle in the middle is composed of 1 + 3 + 5 + 7 = 16 small triangular units. The total area for the triangle of **figure 6** is 12 + 16 + 4 = 32 triangular units.

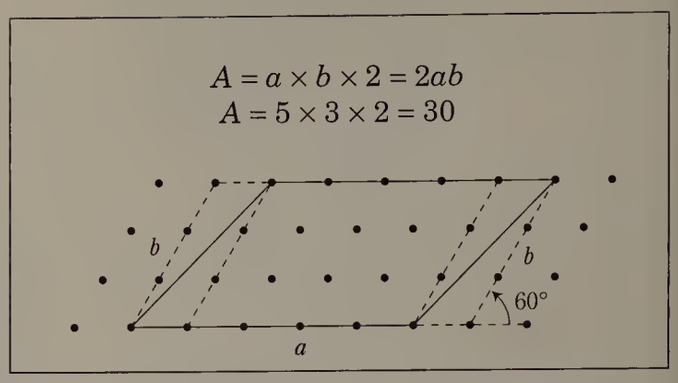


Fig. 7 Using the formula for the area of a parallelogram

With the teacher's cheerleading, this method is designed to awaken a curiosity for devising "formulas" for finding the area of a rectangle, triangle, parallelogram, and trapezoid in triangular units. Students investigate finding areas and formulas for finding areas of these shapes on triangle dot paper using **Activity Sheet 2**.

First, they discover that a parallelogram on the triangle dot paper has a formula for finding its area in triangular units. Given the length of the base, a, and slant height, b, the formula for the area of a parallelogram is $A = base \times slant\ height \times 2$, or $A = 2ab$. The slant height, b, is the length of a segment that makes a 60 degree angle with the base (see **fig. 7**). The partition method leads students to this conclusion as they notice in **figure 7** that the two dotted parallelograms on both sides of the given parallelogram are congruent and are half inside the given parallelogram. They both have area 6, the same as the area of the four parallelograms between them. The area of the given parallelogram is

$$4\times6+\frac{1}{2}(2\times6)=4\times6+1\times6=(4+1)(6)=5\times6.$$

But since each small parallelogram's area is $1 \times 3 \times 2$, where the base is 1 and the slant height is 3, with 2 triangles per level of slant height, the formula for the area of the parallelogram in **figure 7** is

$$\begin{aligned} A &= 5 \times 6 = 5 \times 1 \times 3 \times 2 = 5 \times 3 \times 2 \\ &= base \times slant\ height \times 2 \\ &= a \times b \times 2 = 2ab. \end{aligned}$$

The "Aha!" for students should occur when they realize that the formula is really just a quick way of counting all area "units," in this case, triangular units. On **Activity Sheet 2**, students also explore examples where the slant height may be the same as the length of the side of the parallelogram. (Note: Students should be given an extra sheet of square dot paper and triangle dot paper for investigating additional shapes. These items can be found on various Web sites—use a search engine and look for "square dot paper" or "triangle dot paper.")

$$A = a \times b \times 2 \times \frac{1}{2} = a \times b = 5 \times 3 = 15$$

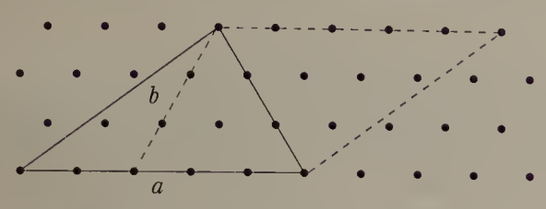


Fig. 8 Using the formula for the area of a triangle

$$A = 2ac - (a - b)(c) = (2a - a + b)(c) = (a + b)(c)$$
$$A = (5 + 2)(3) = (7)(3) = 21$$

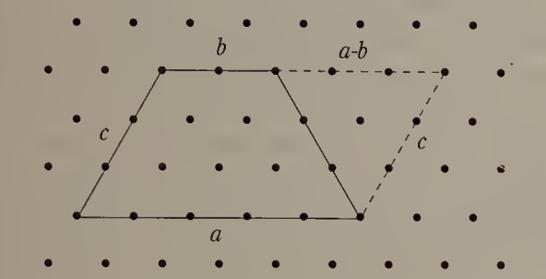


Fig. 9 Using the formula for the area of a trapezoid

Since a rectangle is also a parallelogram, the formula will also hold for rectangles. Thus, returning to **figure 5**, we see that $a = 5$, $b = 2$, and $A = 5 \times 2 \times 2 = 20$.

The formula for the triangle will also be related to the parallelogram since we can always create a parallelogram from the original triangle that has area twice that of the triangle. This is done by constructing the right side of the parallelogram parallel to the left side of the triangle and completing the parallelogram so the top is parallel to the base of the triangle (see **fig. 8**). We can also return to **figure 6** and apply the formula, getting $A = a \times b = 8 \times 4 = 32$.

Finding the area for a trapezoid on triangle dot paper uses the formula for a parallelogram and a triangle as it did on the square dot paper. This is best done by constructing a parallelogram by extending the given trapezoid and using the formula for area of a parallelogram and algebra to find the area of the trapezoid (see **fig. 9**).

After investigating areas on the two types of dot paper, students are asked to look at similarities. They note that the slant height serves the same "role" with the triangular unit that an altitude serves in the standard formulas. Comments by students and teacher observations as they complete the two activity sheets also show that the students have an increased understanding of area and "area" formulas. "We're just counting squares or triangles," one team of four students noted, "but the formulas allow us to do it quicker."

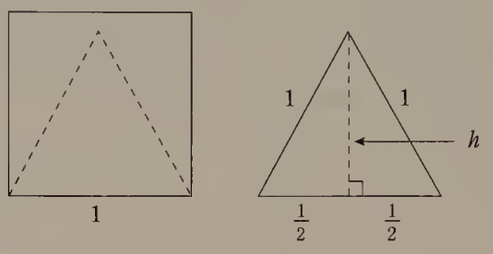
Finally, we might ask, how could we go back and

If the area of the square = 1,

$$A_{triangle} = \frac{1}{2}bh = \frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} \text{ square units}$$

If the area of the triangle = 1,

$$A_{square} = \frac{1}{\frac{\sqrt{3}}{4}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \text{ triangular units}$$



$$h^2 = 1^2 - \left(\frac{1}{2}\right)^2$$
$$h^2 = 1 - \frac{1}{4} = \frac{3}{4}$$
$$h = \frac{\sqrt{3}}{2}$$

Fig. 10 Comparison of square unit and triangular unit

forth between the two types of dot paper? That is, if we have a rectangle on triangle dot paper, what will be its area in square units? For this, we need to find how the triangle unit and square unit compare with each other. That way, if we know the area of the rectangle in triangle units, we can compute its area in square units by using the "conversion" formula (see **fig. 10**).

First, if the 1×1 square is our unit of measure, then we can compute the area of an equilateral triangle with sides of length 1 by applying the formula derived on square dot paper for the area of a triangle: $A = 1/2 \times base \times height = (1/2)bh$. In the case of our "unit" equilateral triangle, $b = 1$, so $b/2 = 1/2$, and using the Pythagorean theorem, $(1/2)^2 + h^2 = (1)^2$. Thus, $h^2 = 1 - 1/4 = 3/4$ and

$$h = \frac{\sqrt{3}}{2}.$$

So, the area of the triangular unit (in square units) is

$$A = \left(\frac{1}{2}\right)(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

(see **fig. 10**).

Conversely, if the equilateral triangle with sides of length 1 is the unit of area, and we want to know the area of the 1×1 square in these triangular units, we are asking, "How many triangular units fit inside the 1×1 square?" Thus, from what we just found, since

$$1 \text{ triangular unit} = \frac{\sqrt{3}}{4} \text{ square units},$$

multiplying both sides by the reciprocal, $\frac{4}{\sqrt{3}}$, gives

$$\frac{4}{\sqrt{3}}(1 \text{ triangular unit}) = \frac{4}{\sqrt{3}}\left(\frac{\sqrt{3}}{4} \text{ square units}\right),$$

or $$\frac{4}{\sqrt{3}} \text{ triangular unit} = 1 \text{ square unit}$$

(see **fig. 10**).

With this relationship between square units and triangular units in mind, we return to **figures 1–9** and list the areas in both systems (see **table 1**). To complete the table, we use a multiplier to go from one area system to the other. If the area in square units is known, we multiply that area by

$$\frac{4\sqrt{3}}{3}$$

TABLE 1

Areas of Figures in Square Units and Triangular Units

Figure	Area in Square Units	Area in Triangular Units
1	12	$12\left(\frac{4\sqrt{3}}{3}\right) = 16\sqrt{3}$
2	$7\frac{1}{2}$	$\left(7\frac{1}{2}\right)\left(\frac{4\sqrt{3}}{3}\right) = 10\sqrt{3}$
3	8	$8\left(\frac{4\sqrt{3}}{3}\right) = \frac{32\sqrt{3}}{3}$
4	9	$9\left(\frac{4\sqrt{3}}{3}\right) = 12\sqrt{3}$
5	$20\left(\frac{\sqrt{3}}{4}\right) = 5\sqrt{3}$	20
6	$32\left(\frac{\sqrt{3}}{4}\right) = 8\sqrt{3}$	32
7	$30\left(\frac{\sqrt{3}}{4}\right) = \frac{15\sqrt{3}}{2}$	30
8	$15\left(\frac{\sqrt{3}}{4}\right) = \frac{15\sqrt{3}}{4}$	15
9	$21\left(\frac{\sqrt{3}}{4}\right) = \frac{21\sqrt{3}}{4}$	21

to get the equivalent area in triangular units. If the area in triangular units is known, we multiply the area by

$$\frac{\sqrt{3}}{4}$$

to get the equivalent area in square units (see **fig. 10**). (Note: This type of investigation does not appear on **Activity Sheet 1** or **Activity Sheet 2** and may be better suited to a geometry or algebra 2 course.)

Is one unit of area preferred over another? It is hoped that this investigation will convince students that for drawings on square dot paper, the square unit is preferred, but for drawings on triangular dot paper, the equilateral triangle unit is not only preferred but there are formulas to aid in computing the area. Fortunately, if needed, we can also express the area of a shape drawn on one type of dot paper in terms of the unit on the other type of dot paper.

Finally, it may be advantageous to draw some shapes on one type of dot paper rather than on the other. For example, it is easy to draw regular hexagons on triangular dot paper although difficult (if not impossible) to draw the shapes on square dot paper (see **fig. 11**).

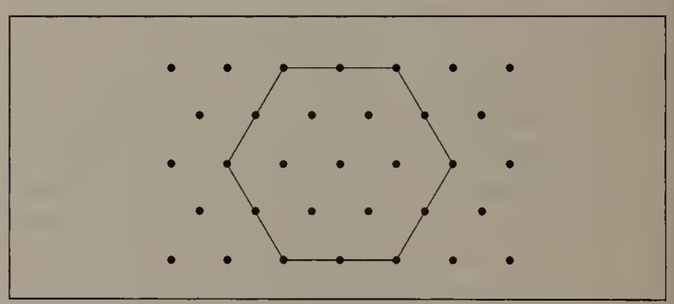


Fig. 11 Regular hexagon on triangle grid paper

By using partitioning, or the formulas for the area of a triangle or the area of a trapezoid, it is easy to find the area of a hexagon in triangular units (24). With our conversion factor of

$$\frac{\sqrt{3}}{4},$$

we can easily find the area in square units ($6\sqrt{3}$). It is left to the students to devise a nice formula for finding the area of a regular hexagon in triangular units.

References

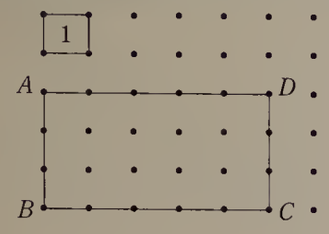
National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

Stigler, James W., and James Hiebert. "Improving Mathematics Teaching." *Educational Leadership* (February 2004): 12–17. □

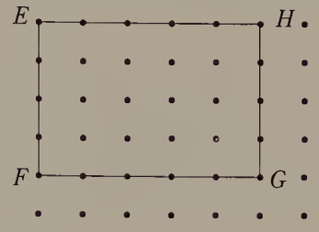
Activity Sheet 1:
Finding Areas on Square Dot Paper

NAME _____

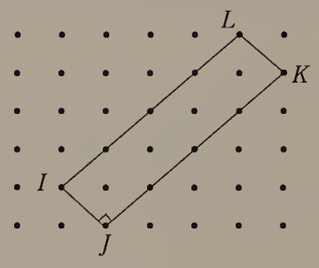
1. Let the unit of area be a 1×1 square. Find the areas of the following rectangles, and use your answers to write the formula for the area of a rectangle with length a and width b. You may use square dot paper to investigate other rectangles and their areas.



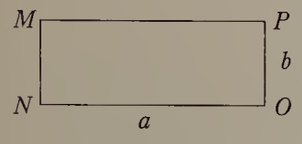
Area of $ABCD$ = $5 \times 3 = 15$
A = $a \times b$



Area of $EFGH$ = _____
A = _____

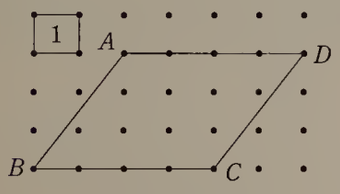


Area of $IJKL$ = _____
A = _____

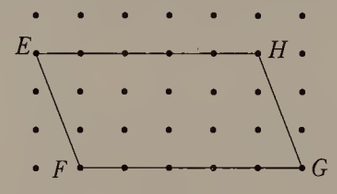


Area of $MNOP$ = _____ A = _____
Explain your reasoning: _____
_____
_____

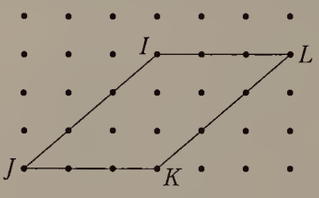
2. Let the unit of area be a 1×1 square. Find the area of the following parallelograms, and use your answers to write the formula for the area of a parallelogram with base a and height b. You may use square dot paper to investigate other parallelograms and their areas.



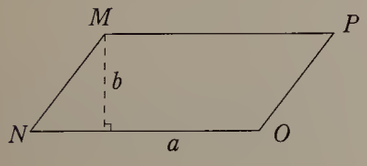
Area of $ABCD$ = _____
A = _____



Area of $EFGH$ = _____
A = _____



Area of $IJKL$ = _____
A = _____



Area of $MNOP$ = _____ A = _____
Explain your reasoning: _____
_____
_____

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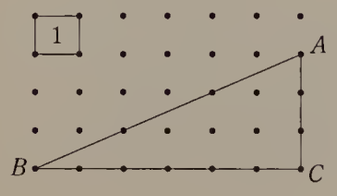
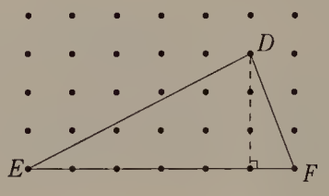
Activity Sheet 1:
Finding Areas on Square Dot Paper *(continued)*

NAME _____

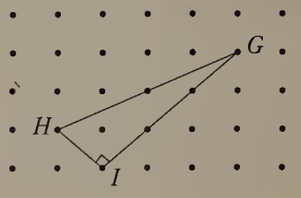
3. Let the unit of area be a 1×1 square. Find the areas of the following triangles, and use your answers to write the formula for the area of a triangle with base a and altitude b. You may use square dot paper to investigate other triangles and their areas.



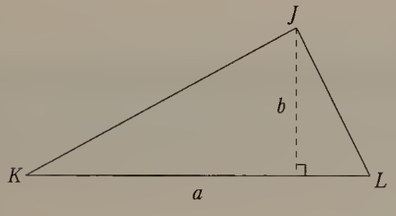
Area of ABC = _____
A = _____



Area of DEF = _____
A = _____

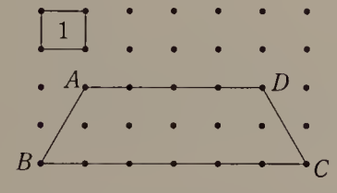


Area of GHI = _____
A = _____

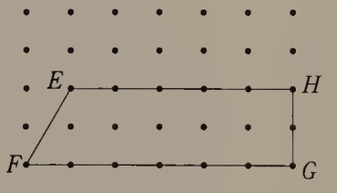


Area of JKL = _____ A = _____
Explain your reasoning: _____
_____
_____

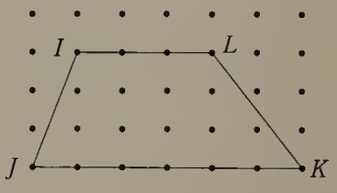
4. Let the unit of area be a 1×1 square. Find the areas of the following trapezoids, and use your answers to write the formula for the area of a trapezoid with bases a and b and height c. You may use square dot paper to investigate other trapezoids and their areas.



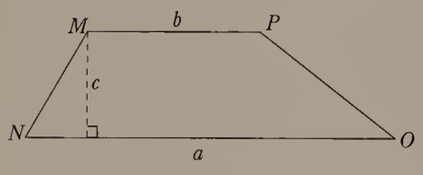
Area of $ABCD$ = _____
A = _____



Area of $EFGH$ = _____
A = _____



Area of $IJKL$ = _____
A = _____



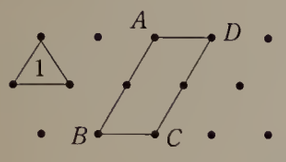
Area of $MNOP$ = _____ A = _____
Explain your reasoning: _____
_____
_____

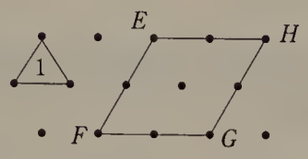
From the December 2006/January 2007 issue of **Mathematics Teaching in the Middle School**

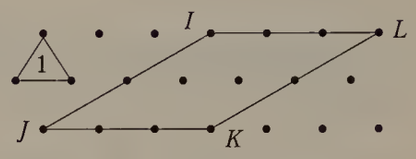
Activity Sheet 2:
Finding Areas on Triangle Dot Paper

NAME _____

1. Let the unit of area be an equilateral triangle with sides of length 1. Find the area of the following parallelograms in triangular units, and use your answers to write the formula for the area of a parallelogram of length a and "slant height" b. You may use triangle dot paper to investigate other parallelograms and their areas.

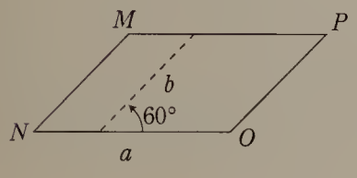






Area of $ABCD$ = _____ Area of $EFGH$ = _____ Area of $IJKL$ = _____

A = _____ A = _____ A = _____



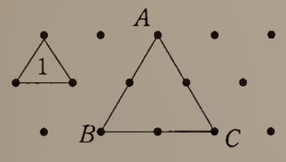
Area of $MNOP$ = _____ A = _____

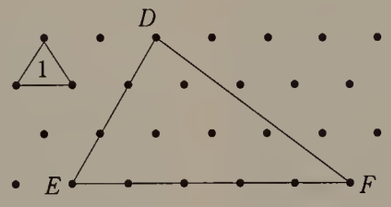
Explain your reasoning: _____

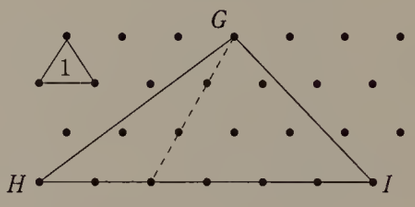
_____

_____

2. Let the unit of area be an equilateral triangle with sides of length 1. Find the area of the following triangles in triangular units, and use your answers to write the formula for the area of a triangle of base a and "slant height" b. (Segment drawn from vertex to the base at a 60° angle.) You may use triangle dot paper to investigate other triangles and their areas.

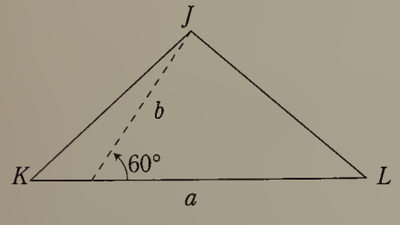






Area of ABC = _____ Area of DEF = _____ Area of GHI = _____

A = _____ A = _____ A = _____



Area of JKL = _____ A = _____

Explain your reasoning: _____

_____

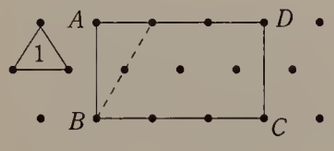
_____

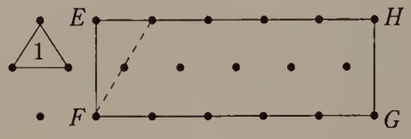
From the December 2006/January 2007 issue of **Mathematics Teaching in the Middle School**

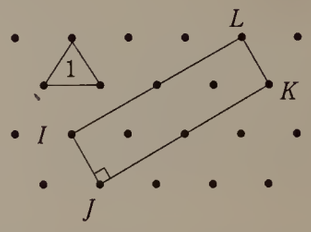
Activity Sheet 2:
Finding Areas on Triangle Dot Paper *(continued)*

NAME _____

3. Let the unit of area be an equilateral triangle with sides of length 1. Find the area of the following rectangles in triangular units, and use your answers to write the formula for the area of a rectangle with base a and "slant height" b. You may use triangle dot paper to investigate other rectangles and their areas.

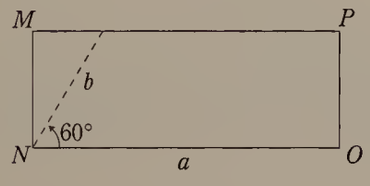






Area of $ABCD$ = _____ Area of $EFGH$ = _____ Area of $IJKL$ = _____

A = _____ A = _____ A = _____



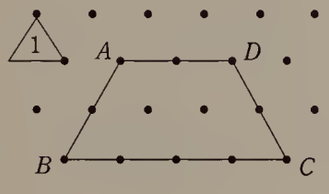
Area of $MNOP$ = _____ A = _____

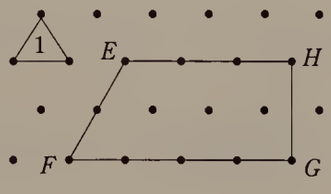
Explain your reasoning: _____

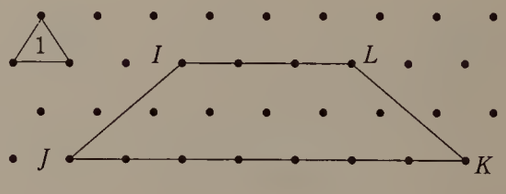
_____

_____

4. Let the unit of area be an equilateral triangle with sides of length 1. Find the area of the following trapezoids in triangular units, and use your answers to write the formula for the area of a trapezoid with bases a and b and "slant height" c. You may use triangle dot paper to investigate other trapezoids and their areas.

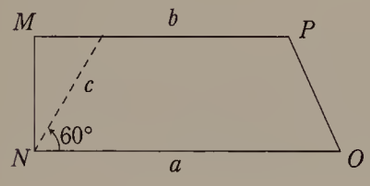






Area of $ABCD$ = _____ Area of $EFGH$ = _____ Area of $IJKL$ = _____

A = _____ A = _____ A = _____



Area of $MNOP$ = _____ A = _____

Explain your reasoning: _____

_____

_____

From the December 2006/January 2007 issue of **Mathematics Teaching in the Middle School**

YEARS of MATHEMATICS TEACHER

HOWARD F. FEHR

Psychology of Learning in the Junior High School

Although we now often use the term "middle school," the educational tenets described in this article remain viable today. Students from the ages of 10 to 15 are prime for meaningful learning. The notion of a general mathematics curriculum containing substantial review of earlier concepts is not acceptable; significant mathematics and problem-solving techniques need to be taught. Just as it was fifty years ago, the role of the classroom teacher is significant in providing motivation and approval.

THE STUDY OF HUMAN BEHAVIOR HAS brought forth an accumulation of facts regarding intelligence that enables us, under proper use, to propose challenging material for instruction. Behavior has many aspects—social, emotional, physical, and mental—and in genuine educational situations all of these aspects must be considered. However, we consider here only those psychological aspects related to intelligence.

It is recognized that the total past experience of a child, the environment in which he has lived and the type of teaching to which he has been subjected before he enters the junior high school, all affect the learning that can and will take place. Nevertheless, from the ages of 11 years to 15 years human beings make continuous and considerable growth in the handling of relationships dealing with space, time, and number, in making comparisons of quantities, in memorization of essential data, in acquisition of new mathematics vocabulary, in using imagination and creative ideas, in abstracting qualities to make generalizations, and in ability to concentrate on the solution of problems. To support these assertions let us look at the findings of research concerning growth of mentality in human beings.

First, learning ability increases at an increasing rate from the ages 11 to 15 years. The *memory span* increases regularly with age. Studies carried on by W. H. Pyle (1)[1] show that the ability to read prose for ideas and to immediately recall these ideas is far greater at age 14 than at age 10 years. Likewise, the ability to look at a configuration (picture) and to recall the items in the picture grows in increasing amounts each year from age 10 to 14. In a study of the time required to learn new words and to memorize numbers the results confirm the fact that each year of age shows a smaller required interval of time. Children learn faster as they grow older from ages 10 to 14. There is no let-up during the junior high school age of both boys and girls in their ability to have an experience in mental activity and recall the experience. In fact this ability grows each year with an increase in storage capacity of the learning and with less time requirement for making it. The experi-

Reprinted from *Mathematics Teacher* 49 (April 1956): 235–40.

A word on the editorial approach to reprinted articles: *Obvious typographical errors have been silently corrected. Additions to the text for purposes of clarification appear in brackets. No effort has been made to reproduce the layouts or designs of the original articles, although the subheads are those that first appeared with the text. The use of words and phrases now considered outmoded, even slightly jarring to modern sensibilities, has likewise been maintained in an effort to give the reader a better feel for the era in which the articles were written.—Ed.*

ments show this to be true whether the material to be learned is meaningful or nonsense, i.e., with relation to previous knowledge or by pure rote. However, Ebbinghaus (2) and others as early as 1880 have shown that meaningful learning is acquired in the least time and has greater permanency for recall.

In learning *perceptual-motor skills* Pyle (1), through a series of studies, found a similar result. Whether it be card-sorting, alphabetizing, mirror drawing, space imagery, or drawing and measuring with instruments, the time required to learn the skill decreases with age. There is no let-up from early age (8 years) until late adolescence (21 years). The age group 10 to 14 years grows in its ability to use tools. The junior high school age should show great growth in the use of rulers, protractors, compasses, and other mathematical instruments.

A higher mental process is that of *grasping relations*. Heidbreder (3) has done outstanding research on the ability of humans to solve a problem by having them discover a principle of recognition through errorless choice. A series of pictures are presented that can be identified as belonging to a particular named class through the recognition of a common property. These pictures are repeated in different forms and sequences until the identification is learned. Children from ages 6 to 10 years, 11 to 15 years, and university students were tested. The rate of learning increased with increasing age. Children grow yearly in their capacity to grasp relations and solve problems.

In connection with this type of learning, studies have been made on the *effect of overlearning*. Overlearning is more thorough learning than is required to perform well on an immediate test. Thus review and drill constitute procedures for overlearning. All experiments prove that overlearning aids greatly in retention and future recall. Review overlearning is more effective when spaced so as to come when some skills are slipping away through lack of use. The best type of review (for retention) appears to be that which incorporates newly learned materials with the old reviewed material. This aids integration and organization of the learning.

Learning ability increases at an increasing rate from the ages 11 to 15 years

As children grow older they change their approach to learning. The *types of learning* assume a higher and more mature form. Children in the elementary school use a trial and error approach to learning that takes much time. In the junior high school the children give greater stress to making analyses of problems and bring about greater reorganization of past experience. (The arithmetic becomes an integrated body of knowledge, not whole numbers and fractions and decimals.) In the senior high school the learning takes on the adult form of circumspection and delayed attacks until the whole problem is studied. This is shown by submitting puzzles to the individuals and watching their manner of solution. The junior high school is the transition stage from immature hit-or-miss trial to the mature analysis and solution-seeking type of learning.

In this transition, transfer of past training is playing a continuously greater role. Under proper instruction, the problem-solving techniques will be headed toward the mature gestalt-insightful learning procedures. The children from ages 11 to 15 years are making this change. They become more cautious, more patient, more systematic, and more organizational in their learning. The teacher is the important guide post. This change in approach to learning has great implications for concept learning and problem solving. It also calls for a rather solid and well-organized knowledge of high school and early collegiate mathematics on the part of the teacher, so that correct mathematical concepts are developed. The transfer and the ability to transfer past knowledge to new situations is dependent on the manner of teaching. "Practice at learning keeps active one's methods of study and yields an increasing body of potentially transferable knowledge and may indirectly increase motivation to learn." Failure to practice learning may be a deterrent to continued learning in the senior high school.

The greatest deterrent to retention and learning is *interference*.[2] Interference is a mixing up of ideas. It occurs when a first or primary idea is not completely understood before proceeding to new ideas or when several unlike ideas are introduced simultaneously. Thus to introduce the solution of equations of the type $2x = 6$ (solved by division) and $(1/2)x = 6$ (solved by multiplication) at the same time may cause interference. At best, if both types of equations are introduced, it would be better to use only one idea, namely, divide by the coefficient of x. As children grow older, a reorganization of responses causes difficulty in learning. Thus, the type of learning that requires the least reorganization is the easiest. In this connection many *new* geometric relations that are first-time experiences are easier to learn than the theory of percent and percentages, where a reorganization of earlier work in arithmetic to new language forms is necessary.

In summary, there is sufficient evidence to prove that there is no let-up from ages 10 to 15 years in growth in mental ability or intelligence as related to memory, perceptual motor skill, concept formation, making of generalizations, and problem solving. Some recent studies on academic achievement have indicated a plateau or leveling off period in acquiring

subject matter is reached during the ages 11 to 14 years. Upon the psychological evidence presented it is the judgment of the writer that this must be ascribed to a nonchallenging curriculum or to sterile teaching and learning situations. The correlations between age and rate of scholastic learning are all over .75 and this means no plateau in academic achievement should exist.

Learning as a Function of Test Intelligence

WE KNOW TODAY THAT IN THE REALM OF ACAdemic study the best indicator of ability is *verbal facility*. The ability to use and understand words, to communicate, is far removed from rote learning. In fact, the data from studies by Thorndike (4), Conrad (5), and Jones (6) on *Observation and Recall as a Function of Age* suggest that *meaning* is the one important dimension determining the correlation between learning and intelligence. *Materials learned under instructions to reproduce meanings* correlate more highly with intelligence than these materials learned by rote. It was also evident that regardless of the type of instruction there is a higher correlation between learning and intelligence to the degree that materials permit attack by a logical method. This relation of meaning and intelligence was sustained in a very recent research by Dr. Max Sobel (7) on learning ninth-grade algebra.

To distinguish between *bright and dull learners* Wilson (8) in 1928 subjected students of high and low intelligence to various types of learning with the following results. *Memory* learning of a multiplication table and shorthand characters yielded the clearest difference. Brightest children have the best memory capacity. Next came reproduction of *meaning* in a paragraph and discovery of a general principle. Simple motor tasks yielded very little difference. Permanent learning of meaningful mathematics is highly correlated with intelligence. Hollingworth (9) in the study of very bright children (IQ 146 to 165) found that the hierarchical order of experiences distinguishing intellectual capacity was: first, abstracting meaning from sentences and paragraphs; second, the study of science; third, learning the meaning of addition of fractions and multiplication of integers. In any learning situation as the number of elements, the complexity, or the difficulty is increased, the difference between good and dull learners became more apparent. Most teachers have discovered this in their own experience. A synthesis of these studies on learning indicate that the five dimensions to learning beginning at the top are: (1) meaning; (2) amount of symbolic process; (3) complexity and difficulty of relations; (4) amount of discovery required; and (5) muscular response. The first four dimensions are significant, but the last is barely related to test intelligence or intelligent action.

Other studies on learning show that a learner can accelerate his *rate of learning* by making a *search for meaning* by the imposition of rhythm or a pattern (by structuring), by making new groupings of items in a situation, by noting spatial relations, and by devising organizations of materials learned. By making a new situation more meaningful he thus assimilates it more readily into his already existent patterns of response. *The "logical method" is far superior to the "rote method" in learning new knowledge.*

The "logical method" is far superior to the "rote method" in learning new knowledge

Motivation

PERHAPS THE MOST MODERN AND THE MOST important psychological aspect of intelligent learning is *motivation based on needs*. Much has been done on this phase of learning both philosophically and experimentally, and the literature is vast. We shall report here only that which appears significant for learning in the junior high school. Symonds (10) has given an interpretation of Maslow's hierarchical order of needs of human beings as follows:

a. Objects and persons to satisfy inner psychological needs
b. Safety—escape from outer dangers
c. Acceptance, involving affection, affiliation, belonging, companionship
d. Approval, prestige, status
e. Self-esteem, self-respect, self-satisfaction
f. Mastery, success, achievement
g. Independence

The first three needs are basic early in life. It is the last four needs that rise in junior high school boys and girls. All these needs involve the self, that is, the individual as a person. Whatever satisfies these needs becomes the goal of the individual. Motivation is goal-directed behavior. To motivate a pupil, present him with goals to be attained, which when attained will satisfy the needs listed in "d" to "g" above. If we accept these needs as basic, then the self of the student is most important. Motivation for most human learning resides in the inter-personal relationship of the student with other students and with his superiors (teachers and parents). The most basic incentives to learning a teacher can give are "acceptance," "approval," and "respect." Travel from convention to convention and seek the reason for the great things mathematicians are doing, and

you will find that mastery, success, independence, recognition, and approval are the greatest drives to learning and creating—not simply the love of mathematics itself, as great as this may be.

The most basic incentives to learning a teacher can give are "acceptance," "approval," and "respect"

A pupil in junior high school will continue to learn or to be discouraged from learning to the degree that his expectations or goals have been reached. The influence of success is a tremendous incentive. An experiment has been performed in which students were given problems to be solved and then, after a given time, urged to solve the next one. Some of the problems were not possible of solution by the students. Success led to heightened interest and eagerness to do the next one. The reaction to immediate failure was hard to estimate, but when three or four insolvable problems occurred in a row, there was practically no incentive to continue. It thus appears that intermittent rewards (success) seem most favorable to motivating further learning. Developing self-confidence is an incentive to learning. Best of all, enthusiasm of the teacher for his subject seems to be a major factor in motivating his students to learn.

Competition has long been held one of the key incentives to further learning. Recent investigations throw much doubt on this hypothesis. Among the major findings are the following (11):

1. It violates the agreement that learning is possible only if the goal is attainable. When a whole class of heterogeneous students compete for the top prize, there is motivation only for the very best in the class, for only the best can attain the prize or reward by winning; for the others there is only defeat. Thus competition, if used continuously, makes the strong stronger in learning, and the weak, who need more incentive than the strong, benefit not at all.
2. Competition for high honors or even just to pass a high standard is not a genuine learning situation, in that the purpose is to excel and not necessarily to understand one's own processes. Only the excellent get the approval and self-esteem so necessary for learning. It damages the self-respect of the losers and interferes greatly with motivation for further learning.
3. Good competition is with one's own peers or with one's own prior achievement. This is competition that permits success as well as failures. It has been shown that continued success without any failure causes a lack of motivation as well as continued failure. Self-esteem comes with real achievement, achievement that has passed by difficult hurdles and not merely followed a rosy path with no troubles.

Along with competition, school grades, or marks, have long been one of the teacher's means of motivation. Now a high grade, A or 100, is frequently motivation to learning, not because of rewards, but because of the more basic satisfaction of approval, status, and success. We must remember that basic motivations reside in human relations, and that the striving for external rewards such as high marks or honors is subordinate to and hidden in the striving for direct recognition and approval. The boy or girl interprets a high grade to mean "you approve," and by the same token he interprets a low grade to mean "you disapprove" of him, you have cast him out, he has lost face. You may try very hard to tell these boys and girls that they get only what they earn, and the mark is what they have earned, yet the boys and girls will not listen to you. Yes, good grades motivate, but low grades discourage and do not motivate, and in neither case do the grades show how learning should take place. To be admired, approved, and accepted are basic to motivating learning.

Comparing the psychology of learning with the present curriculum in junior high school mathematics must raise some serious problems in our minds. Certainly we should have more challenging intellectual material because our pupils are not achieving to capacity, especially the more able students. Certainly we can well consider our methods of teaching, especially in light of the necessity and possibility of realizing better reading, understanding, and problem-solving ability. There would also be quite some doubt as to the procedures we are using to motivate and keep alive a burning desire to study mathematics, a very much needed goal of our future citizenry. The most immediate problem appears to be that of creating a body of mathematics, knowledge of sufficient complexity and abstractness to be commensurate with the growing intellectual capability of the students.

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Endnotes

1. The numbers in parentheses refer to the bibliography at the end of the article.
2. When a student's attention is distracted so that he is not attending the instruction, we do not think of this as interference. This is removal from the learning situation. □

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by January 31, 2007, and answer questions about the content of the journal.

In the course of taking the survey, you will be asked to identify yourself. You need not do so, but participants who identify themselves will be entered in a lottery for a $50 NCTM gift certificate. Your identifying information will only be used for the purposes of contacting you should your name be chosen in the lottery.

Mathematics Detective: Teacher Notes and Solutions

(Continued from pages 260–61)

An Excel spreadsheet for teachers, listing plans and costs, is shown below.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3											
4											
5											
6		Month	Daytime	Overage	Cost for	Nights	Total Minutes	Overage	Cost for	Nights	Weekend
7			(7am-6:59pm)	Plan E	Plan E	(7pm-8:59pm)	Plan F	Plan F	Plan F	(9pm-6:59am)	(9pm-6:59am)
8			(M-F)			(M-F)				(M-T)	(F-M)
9		1	652		144.9	163	815	115	151.71	63	211
10		2	700		144.9	108	808	108	148.56	71	264
11		3	637		144.9	195	832	132	159.36	42	196
12		4	720	20	152.9	94	814	114	151.26	65	147
13		5	586		144.9	137	723	23	110.31	56	214
14		6	681		144.9	118	799	99	144.51	37	201
15		7	725	25	154.9	65	790	90	140.46	49	183
16		8	556		144.9	244	800	100	144.96	68	238
17		9	697		144.9	112	809	109	149.01	57	119
18		10	740	40	160.9	51	791	91	140.91	53	175
19		11	732	32	157.7	115	847	147	166.11	55	183
20		12	588		144.9	253	841	141	163.41	66	164
21	Averages			9.75	148.8			105.75	147.5475		
22	Averages			148.9				147.66			

1.

Month	Plan E	Plan F
Month 1	$144.90	$151.71
Month 2	$144.90	$148.56
Month 3	$144.90	$159.36
Month 4	$152.90	$151.26
Month 5	$144.90	$110.31
Month 6	$144.90	$144.51
Month 7	$154.90	$140.46
Month 8	$144.90	$144.96
Month 9	$144.90	$149.01
Month 10	$160.90	$140.91
Month 11	$157.70	$166.11
Month 12	$144.90	$163.41

2. One way to solve this question is to average the number of minutes over 700 for each plan (round to the nearest whole minute), multiply appropriately, and add to the basic plan per month. If this method is used, plan F is cheaper at $147.66, whereas plan E costs $148.90. Another solution method is to calculate each month's bill, then average. If this is used, plan F remains the more affordable option at $147.55 versus $148.80 for plan E.

3. It seems to make the most difference when the number of daytime minutes is below 700 and the number of minutes used between 7:00 p.m. and 8:59 p.m. is 100 or more. This results in a higher bill for Flash Cellular. When the number of daytime minutes is over 700, Express Wireless is usually higher. □

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- Supporting English as Second Language learners
- Reaching low- and high-achieLving students
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- Connecting to assessment

Examples of ways that teachers have addressed representations in the classroom and how students have responded to the use of varied representations are requested. Suggested topics for a manuscript include, but are not limited to, articles that address one or more of the following questions:

- What representations are most useful for a particular mathematical concept?
- Are specific mathematical concepts or ideas taught or learned more effectively through the use of multiple representations?
- How does flexibility in the use of representations impact student understanding, and does it vary across grade levels?
- What are barriers or challenges in teaching through representations?
- How does the use of technology relate to an effective use of representations?
- How does the use of representations impact English as Second Language learners, gifted students, or students with specific learning disabilities?
- What resources (print, manipulative, or electronic) are most useful in teaching specific mathematical ideas through varied representations?
- How are representations used to differentiate instructions?
- How are representations used to assess students' mathematics understanding?
- Why do different students understand certain representations better than others?
- How do representations support connections, reasoning, communication, and problem solving?
- What is the role of the teacher regarding the use of representations in the middle-grades classroom?
- What has classroom action research shown about students' learning of mathematics through representation?
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The manuscript should not exceed ten double-spaced pages of text, and figures and photographs should be included at the end. Send submissions to this call for manuscripts by accessing **mtms.msubmit.net**. When submitting online, scroll down to "Manuscript Type" and click on "Focus Issue." The due date is **February 15, 2007**. □

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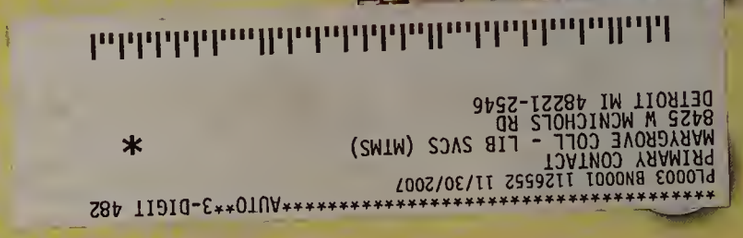

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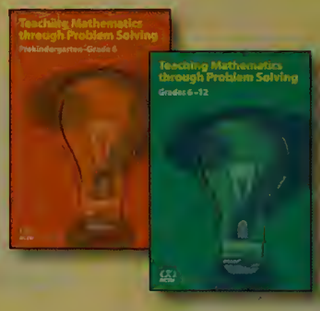

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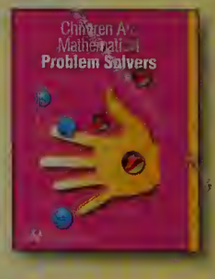
Children Are Mathematical Problem Solvers



This book comprises a range of problems from the Problem Solvers column in *Teaching Children Mathematics*, NCTM's journal for elementary school teachers. The situations that the problems offer engage students in explorations in which they do challenging, interesting problem solving with significant mathematical content.

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